

MAXIMAL r -DIAMETER SETS AND SOLIDS OF CONSTANT WIDTH

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Abstract: We recall the definition of an r -maximal set in a metric space as a maximal subset of diameter r . In the special case when the metric space is Euclidean such a set is exactly a solid of constant diameter r . In the process of reviewing the theory of these objects we provide a simple construction which generates a large class of such solids.

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INTRODUCTION

I was reading a paper - reviewing it actually - in which a subset was being covered by sets of diameter at most r for some fixed $r > 0$. It occurred to me that if a subset is rather elongated then the diameter constraint is not completely binding in the sense that we can enlarge the set laterally without increasing the diameter. This suggests the question, which can be asked in any metric space X , of characterizing the subsets of diameter r which are maximal with respect to this condition.

Eventually I discovered that for Euclidean space this notion is quite old. Under the name *completeness* or *diametrical completeness* the concept was introduced by Meissner in 1909 and its relation with the related concept of *constant width* has been the object of considerable

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study. Most of what I rediscovered had been analyzed by Minkowski and was described in Bonnesen and Fenchel's (1934) survey, see also Eggleston (1958) and Lay (1987). We will follow Eggleston (1965) in using *diametrical maximality* in place of the overused term *completeness*. We also overlap recent work in Lachand-Robert and Oudet (2007) and Bayen, Lachand-Robert and Oudet (2007).

From Zorn's Lemma it follows that any set of diameter at most r , which we will call an r -bounded set, is a subset of some maximal subset of diameter r . We will call such a set an r -maximal set.

In an r -bounded set C we will call a pair $x_1, x_2 \in C$ *antipodal* if $d(x_1, x_2) = r$.

Conditions on r -bounded sets depend on the underlying metric. Call the metric d *open* (or *proper*) when for all $x \in X$ the function from X to $[0, \infty)$ defined by $y \mapsto d(x, y)$ is an open map (resp. a proper map). Thus, d is proper iff any closed, finite diameter subset is compact. For any compact subset C of X define $d_+(C, x) = \max\{d(x, y) : y \in C\}$. Equivalently, $d_+(C, x)$ is the radius of the smallest closed ball centered at x which contains C . Call d *connected* when for every compact C and any $r > 0$ the open set $\{x \in X : d_+(C, x) < r\}$ is connected and, when it is nonempty, its closure is $\{x \in X : d_+(C, x) \leq r\}$.

Theorem 0.1. *Let X be a metric space with a proper, open metric.*

(a) *If C is an r -maximal subset then C is compact and for every x_1 in the topological boundary of C there is an antipodal point x_2 in the boundary of C . Furthermore, $C = \{x \in X : d_+(C, x) \leq r\}$ and the interior of C is $\{x \in X : d_+(C, x) < r\}$. If the metric is connected then C is connected.*

(b) *Assume that the metric is connected and that C is a compact r -bounded subset with nonempty interior. If every point of the boundary of C has an antipode in C then C is r -maximal.*

If E is a Minkowski space, i.e. a finite dimensional, normed linear space then the associated metric is proper, open and connected. For a functional ω in the unit sphere S^* of the dual space of E we define the ω diameter of a subset C to be $\sup\{\omega(x_1 - x_2) : x_1, x_2 \in C\}$. We say that C has *constant diameter* r if the ω diameter of C equals r for every $\omega \in S^*$.

Theorem 0.2. *Let E be a Minkowski space.*

(a) *If C is an r -maximal subset then C is a compact convex set with nonempty interior.*

(b) *If C is a compact, convex set of constant diameter r then C is an r -maximal subset.*

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By a Euclidean space we mean a finite dimensional, linear space with norm obtained from an inner product. When the dimension is n such a space is isometric to \mathbb{R}^n with the usual metric.

Theorem 0.3. *Let E be a Euclidean space.*

(a) *A compact convex subset C is r -maximal iff it has constant diameter r .*

(b) *If a point x of an r -maximal set C has two distinct antipodal points y_1, y_2 in C then the arc between them centered at x is entirely contained in the boundary of C . Conversely, if an arc of radius r is contained in the boundary of an r -maximal set C then the center x lies in the boundary of C and for each point y of the arc other than the endpoints x is the unique point of C antipodal to y .*

In the Euclidean case there is a simple construction which yields a large class of examples and which appears to be new.

Let g be a C^2 real-valued function on the unit sphere S in \mathbb{R}^n ($n \geq 2$). Assume that g is an odd function, i.e. $g(-u) = -g(u)$ for all $u \in S$. Extend g to a C^2 odd, homogeneous function of degree 1 defined on $\mathbb{R}^n \setminus 0$ by $g(x) = |x|g(x/|x|)$. Let $H = (h_1, \dots, h_n) : \mathbb{R}^n \setminus 0 \rightarrow \mathbb{R}^n$ be the gradient of g so that $h_i = \partial g / \partial x_i$ for $i = 1, \dots, n$. Each h_i is a C^1 even, homogeneous function of degree 0. We say that g satisfies the *r -Maximality Condition* when the Jacobian matrix $(\partial^2 g / \partial x_i \partial x_j)$ has all eigenvalues contained in $[-(r/2), (r/2)]$ at every point u of S . Notice that for g any C^2 odd function on S there is a maximum positive λ^* such that λ^*g satisfies the *r -Rotundity Condition* and then λg satisfies *r -Maximality Condition* for all $\lambda \in [0, \lambda^*]$.

Theorem 0.4. *If g satisfies the r -Maximality Condition then the subset of \mathbb{R}^n*

$$\begin{aligned} C &= \{(H(u) + t(r/2)u) : (u, t) \in S \times [-1, 1]\} \\ &= \{(H(u) + t(r/2)u) : (u, t) \in S \times [0, 1]\} \end{aligned}$$

is a solid of constant diameter r , and so is an r -maximal set.

In general, if C is a solid of constant width r then for each $u \in S$ there is a unique directed segment of length r with direction u connecting a pair of antipodal points in C . We can define $H(u)$ is the midpoint of the segment. This is called the *median surface* by Bayen, Lachand-Robert and Oudet (2007), see also Guilfoyle and Klingenberg (2009). The function H is even, i.e. $H(-u) = H(u)$, because the segment for $-u$ connects the same pair but with the orientation reversed. This explains why the two descriptions of C above yield the same set.

Letting λ vary in $[0, \lambda^*]$ we obtain a one parameter family of r -maximal sets connecting C to the unit ball centered at 0.

In the planar case, a related construction shows that the radius of curvature is the only constraint on including a piece of a curve in the boundary of some r -maximal set.

Theorem 0.5. *Given a plane curve with radius of curvature bounded by r at every point then any sufficiently short piece can be embedded in the boundary of some r -maximal subset of the plane.*

1. GENERAL METRIC SPACES

We recall the language of relations, see e.g. Akin (1993). For sets X and Y a *relation* R from X to Y is an arbitrary subset $R \subset X \times Y$. For $x \in X$, $R(x) = \{y \in Y : (x, y) \in R\}$ and for $A \subset X$, $R(A) = \bigcup_{x \in A} R(x)$. The inverse relation R^{-1} from Y to X is $\{(y, x) : (x, y) \in R\}$. The relation R is called *surjective* when $R(X) = Y$ and $R^{-1}(Y) = X$. That is, R projects onto each factor. When $X = Y$ we call $R \subset X \times X$ a relation on X . If X and Y are topological spaces then R is a closed (or open) relation when it is a closed (resp. open) subset of the product.

Now let X be a metric space with metric d . With $r > 0$ define relations on X

$$\begin{aligned} V_r &= \{(x, y) \in X \times X : d(x, y) < r\}, \\ \bar{V}_r &= \{(x, y) \in X \times X : d(x, y) \leq r\}, \\ A_r &= \{(x, y) \in X \times X : d(x, y) = r\}. \end{aligned} \tag{1.1}$$

For $C \subset X$ the diameter $\text{diam}(C) = \sup\{d(x, y) : x, y \in C\}$. Thus, $\text{diam}(C) \leq r$ iff $C \times C \subset \bar{V}_r$. Since \bar{V}_r is closed it follows that $\text{diam}(C) = \text{diam}(\bar{C})$ where \bar{C} is the closure of C . C is called *bounded* when it has finite diameter.

Let $\text{Bdry}(C)$ denote the topological boundary of C so that $\text{Bdry}(C) = \bar{C} \setminus C^\circ$, where C° is the interior of C .

For a nonempty subset $C \subset X$ and a point $x \in X$ we define

$$\begin{aligned} d_-(C, x) &=_{\text{def}} \inf\{d(y, x) : y \in C\}, \\ d_+(C, x) &=_{\text{def}} \sup\{d(y, x) : y \in C\}. \end{aligned} \tag{1.2}$$

Of course, $d_+(C, x)$ is only finite when C is bounded.

For $C \subset D$ it is clear that

$$d_-(D, x) \leq d_-(C, x) \leq d_+(C, x) \leq d_+(D, x). \tag{1.3}$$

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For any subset $D \subset X$,

$$(1.4) \quad \sup\{d_-(C, x) : x \in D\} = \inf\{r \geq 0 : D \subset \bar{V}_r(C)\}.$$

Of course, if C and D are compact then all of these sups and infs are achieved.

For compact subsets C, D the *Hausdorff distance* is defined to be

$$(1.5) \quad d(C, D) =_{\text{def}} \min\{r \geq 0 : D \subset \bar{V}_r(C) \text{ and } C \subset \bar{V}_r(D)\}.$$

For properties of the space of compact subsets equipped with the Hausdorff metric see Akin (1993) Chapter 7 or Kuratowski (1968) Section 42.

From the triangle inequality it easily follows that for nonempty compacta $C, D \subset X$ and points $x, y \in X$

$$(1.6) \quad \begin{aligned} |d_-(C, x) - d_-(D, y)| &\leq d(C, D) + d(x, y); \\ |d_+(C, x) - d_+(D, y)| &\leq d(C, D) + d(x, y). \end{aligned}$$

For any bounded subset C of X we define the *antipodal relation* A_C to be the closed, symmetric relation on C

$$(1.7) \quad A_C = A_r \cap (C \times C) \quad \text{where} \quad r = \text{diam}(C).$$

Call the metric d *proper* when for each $x \in X$ the function $d(x, \cdot)$, i.e. $y \mapsto d(x, y)$ is a proper map from X to $[0, \infty)$. That is, the pre-image of a compact set is compact. Equivalently, d is proper when any closed, bounded subset is compact. Call d *open* when for each $x \in X$ the function $d(x, \cdot)$ is an open map from X to $[0, \infty)$. Notice that when the metric is proper, X is locally compact and if the metric is open then X is not compact.

We can localize the rather strong condition that the metric be open. Call $r > 0$ a *regular value* for $x \in X$ when r is neither a local maximum or a local minimum value for $d(x, \cdot)$. That is, if $d(x, y) = r$ then y is neither a local maximum point, nor a local minimum point for $d(x, \cdot)$. Call r a regular value for $C \subset X$ when it is a regular value for every $x \in X$. When the metric is open, $d(x, \cdot)$ has no local maxima or positive local minima and so every positive r is regular for every $x \in X$.

Lemma 1.1. *Let C be a subset of X with $\text{diam}(C) = r$ and let $x \in C$.*

(a) *If $x, y \in C$ and r is a regular value for y then $d(x, y) = r$ implies $x \in \text{Bdry}(C)$.*

(b) *If $x \in C^\circ$ and r is a regular value for C then $A_C(x) = \emptyset$.*

Proof: (a): Since r is a regular value for y there exist $z \in X$ arbitrarily close to x with $r < d(z, y)$. Since $\text{diam}(C) = r, z \notin C$. Hence, $x \notin C^\circ$.

(b): Apply (a) with $y \in C$ arbitrary.

□

Definition 1.2. For $r > 0$ and a subset C of X . We will call C r -bounded when the diameter of C is at most r . C is called a diametrically maximal set of size r , or just r -maximal, when C is a maximal r -bounded subset of X .

For C a nonempty subset of X and $r > 0$ define the r -dual

$$(1.8) \quad \begin{aligned} C_r^* &=_{\text{def}} \{y \in X : C \subset \bar{V}_r(y)\} \\ &= \bigcap \{\bar{V}_r(x) : x \in C\} = \{d_+(C, \cdot) \leq r\}, \end{aligned}$$

where we use the notation $\{d_+(C, \cdot) \leq r\}$ for $\{y \in X : d_+(C, y) \leq r\}$ and similarly for $\{d_+(C, \cdot) < r\}$.

Theorem 1.3. Let $C \subset X$.

- (a) For every $r > 0$, C_r^* is a closed subset of X .
- (b) If $C \subset D$ then $D_r^* \subset C_r^*$.
- (c) $C \subset C_r^*$ if and only if C is r -bounded.
- (d) If C is r -bounded then

$$(1.9) \quad C_r^* = \{y \in X : C \cup \{y\} \text{ is } r\text{-bounded}\}.$$

- (e) C is contained in an r -maximal set if and only if C is r -bounded.
- (f) C is an r -maximal set if and only if $C = C_r^*$.
- (g) If C is r -bounded then the following are equivalent:
 - (i) C is contained in a unique r -maximal set.
 - (ii) C_r^* is an r -maximal set.
 - (iii) C_r^* is r -bounded.

When these conditions hold, C_r^* is the unique r -maximal set containing C .

Proof: (a), (b) and (c): Obvious.

(d): If $\text{diam}(C) \leq r$ then $\text{diam}(C \cup \{y\}) \leq r$ iff $d(y, x) \leq r$ for all $x \in C$ and so iff $y \in C_r^*$.

(e): An r -round set is r -bounded and so any subset is r -bounded. On the other hand, the condition $C \times C \subset \bar{V}_r$ is a property of finite type. So it follows from Zorn's Lemma that any r -bounded subset C is contained in a maximal r -bounded set.

(f): If C is r -maximal or if $C = C_r^*$ then C is r -bounded. Clearly, C is r -maximal iff $C = \{y \in X : \text{diam}(C \cup \{y\}) \leq r\}$ and so iff $C = C_r^*$ by (d).

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(g): Observe first that by (b) and (f)

$$(1.10) \quad C \subset D \quad \text{and} \quad D \text{ is } r\text{-round} \quad \implies \quad D = D_r^* \subset C_r^*.$$

(i) \Rightarrow (ii): Let D be the unique r -maximal set containing C and let $y \in C_r^*$. By equation (1.9) and (e) $\{y\} \cup C$ is contained in some r -maximal set which must be D by uniqueness. Hence, $y \in D$. As y was arbitrary $C_r^* \subset D$. From implication (1.10) it follows that $C_r^* = D$ and so is itself r -maximal.

(ii) \Rightarrow (iii): An r -maximal set is r -bounded.

(iii) \Rightarrow (i): If D is an r -maximal set which contains C then by (1.10) again $D \subset C_r^*$. Since C_r^* is r -bounded by assumption, maximality of D implies $D = C_r^*$. Hence, C_r^* is the unique r -maximal set which contains C .

□

Definition 1.4. Let C be an r -bounded subset of X . We say that C satisfies the antipodal condition if for every $x \in \text{Bdry}(C)$ there exists $y \in C$ such that $d(x, y) = r$, i.e. y is antipodal to x . Equivalently, $A_C(x) \neq \emptyset$ for all $x \in \text{Bdry}(C)$.

Thus, if r is a regular value for C , e.g. if the metric is open, then C satisfies the antipodal condition iff A_C is a surjective relation on $\text{Bdry}(C)$.

Proposition 1.5. If C be an r -maximal subset of X then C is a closed subset of X . If d is proper then C is compact.

Proof: $C = \overline{C}$ because taking closure does not increase diameter. Alternatively, apply Theorem 1.3(a) and (f). If d is proper then C is compact because it is closed and bounded.

□

Theorem 1.6. Assume that d is proper. If $\{C_n\}$ is a sequence of r -maximal sets which converges to a compact set C with respect to the Hausdorff metric, then C satisfies the antipodal condition. In particular, an r -maximal set satisfies the antipodal condition.

Proof: Because the metric is proper and the sequence $\{C_n\}$ is convergent, the set $X_0 = \overline{\bigcup_n C_n}$ is closed and bounded and so is compact. Furthermore, the limit C is the *limsup* of the sequence. That is,

$$(1.11) \quad C = \bigcap_n \overline{\bigcup_{k \geq n} C_k}.$$

See Akin (1993) Lemma 7.5.

Fix $x \in \text{Bdry}(C)$ and let $\{x_n\}$ be a sequence in $X \setminus C$ which converges to x .

For each n there exists $i_n \geq n$ such that $x_n \in X \setminus C_{i_n}$. Because C_{i_n} is r -maximal maximality implies that $\{x_n\} \cup C_{i_n}$ has diameter larger than r and so there exists $y_n \in C_{i_n}$ for each n such that $d(x_n, y_n) > r$. Let y be a limit point of this sequence in X_0 . Because $i_n \rightarrow \infty$ the point y is in the $\text{limsup} = C$. Furthermore, $d(x, y) \geq r$. Since $x, y \in C$ we have $d(x, y) \leq r$ and so x and y are antipodal.

If C is r -maximal then it is the limit of the constant sequence $C_n = C$ of r -maximal sets.

□

Corollary 1.7. *Let C be a compact subset of X . Assume that $r > 0$ is a regular value for C (as is always true when the metric is open). subset of X .*

(a) *The interior of $C_r^* = \{d_+(C, \cdot) \leq r\}$ is $\{d_+(C, \cdot) < r\}$.*

(b) *If the metric is proper and C is an r -maximal subset of X then A_C is a closed, symmetric surjective relation on the compact set $\text{Bdry}(C)$. Furthermore,*

$$(1.12) \quad C^\circ = \bigcap \{V_r(x) : x \in C\} = \{d_+(C, \cdot) < r\}.$$

Proof: (a): Both C and its boundary are compact. Let $y \in C$. Because r is a regular value for y , $V_r(y)$ is the interior of $\bar{V}_r(y)$ and the boundary of each of these sets is $A_r(y)$. The open set $\{d_+(C, \cdot) < r\}$ is contained in the interior of $\{d_+(C, \cdot) \leq r\} = C_r^*$. Conversely, if x is a point of the interior of C_r^* then it is in the interior of $\bar{V}_r(y)$ which is $V_r(y)$. Thus, $x \in V_r(y)$, for every $y \in C$. By compactness of C , $d_+(C, x) < r$.

(b): A_C is always symmetric and closed. If $x \in \text{Bdry}(C)$ then by Proposition 1.6 there exists $y \in A_C(x)$ and by Lemma 1.1, $y \notin C^\circ$. Hence, A_C is a surjective relation on $\text{Bdry}(C)$.

Equation (1.12) follows from (a) because $C = C_r^*$.

□

Call the metric d *connected* if for every nonempty compact set $C \subset X$ and every $r > 0$ the open set $\{d_+(C, \cdot) < r\}$ is connected and if it is nonempty then its closure is $\{d_+(C, \cdot) \leq r\}$. In particular, for any $x \in X$ and $r > 0$ the open set $V_r(x)$ is open and connected with closure $\bar{V}_r(x)$. Hence, the latter is a regular closed set which is connected. Recall that a closed set is called a *regular closed set* when it is the closure of an

open set and so is the closure of its interior. If r is a regular value for x then $V_r(x)$ equals the interior of $\bar{V}_r(x)$.

Lemma 1.8. *Let X have a connected metric d and let C be a compact subset of X . Assume that $r > 0$ is a regular value for C . If C_r^* has a nonempty interior then the interior equals $\{d_+(C, \cdot) < r\}$ and it is a connected open set. C_r^* is a connected, regular closed set.*

Proof: By Corollary 1.7(a) the interior of C_r^* is $\{d_+(C, \cdot) < r\}$ which is connected and nonempty by assumption. Hence, $C_r^* = \{d_+(C, \cdot) \leq r\}$ is the closure of its interior. Since the closure of a connected set is connected C_r^* is connected.

□

Theorem 1.9. *Let X be a space with a proper, connected metric, let C be a subset of X . Assume that $r > 0$ is a regular value for C , as is always true when the metric is also open.*

(a) *If C is r -maximal and has a nonempty interior then the interior is connected and C is a connected, regular closed set.*

(b) *If C is r -maximal then it is a compact connected set.*

(c) *If C is a closed r -bounded subset with a nonempty interior which satisfies the antipodal condition, then C is r -maximal.*

Proof: (a): Since the metric is proper, C is compact and it equals C_r^* by Theorem 1.3 (f). So the result follows from Lemma 1.8.

(b): For any C let $e > 0$ and $r_e = r + 2e$ and let C_e be an r_e -round set containing the r_e -bounded set $\bar{V}_e(C)$.

Claim: For every $\epsilon > 0$ there exists a $\delta > 0$ such that $e < \delta$ implies $C_e \subset V_\epsilon(C)$.

Proof: If there exists $\epsilon > 0$ and a sequence of positive e_n 's converging to 0 and $x_n \in C_{e_n}$ with $d(x_n, C) > \epsilon$ then by going to a subsequence we obtain a limit point x not in C but such that the diameter of $C \cup \{x\}$ is r . This contradicts r -rotundity of C .

Thus, we can inductively choose a sequence of C_{e_n} 's with $C_{e_{n+1}} \subset C_{e_n}^\circ$ and with intersection C . As each C_e has a nonempty interior and is r_e round, it is compact and connected by (a). Hence, the intersection C is compact and connected.

(c): Let D be an r -maximal set which contains C and so has a nonempty interior. By (a) D is the closure of its interior and its interior is connected. Every point x of the boundary of C has an antipodal point y in C . That is $d(x, y) = r$. Since r is a regular value for y , we can apply Lemma 1.1(a) with C replaced by D to see that $x \in \text{Bdry}(D)$. Since the boundary of D is disjoint from the interior of D , the interior

of C is nonempty -by hypothesis- and is clopen in the interior of D . As the latter is connected, the interior of C equals the interior of D . Since D is a regular closed set it must be contained in the closed set C and so $D = C$. Thus C is r -maximal.

□

I do not know whether an r -maximal set with empty interior can exist when the metric is proper, open and connected. We do have the following characterization of such odd cases.

Proposition 1.10. *Let X have a proper, open, connected metric and let C be an r -maximal set with empty interior. We have $d_+(A_C(x), y) \geq r$ for every $x \in C$ and every $y \in X$. In particular, $A_C \circ A_C(x) = C$ for every $x \in C$.*

Proof: Let U be a closed neighborhood of $A_C(x)$ in C . Assume that $\{d_+(U, \cdot) < r\}$ is nonempty. Then its closure is $\{d_+(U, \cdot) \leq r\}$ which contains all of C and, in particular, contains x . $d_+(C \setminus U^\circ, x) < r$ and so by continuity we can choose $y \in X$ with $d_+(U, y) < r$ and close enough to x that $d_+(C \setminus U^\circ, y) < r$. Thus, $d_+(C, y) < r$. Since C is r -maximal, (1.12) implies that y is in the interior of C but we have assumed that $C^\circ = \emptyset$.

The contradiction implies that for each $y \in X$ and all closed neighborhoods U of $A_C(x)$, $d_+(U, y) \geq r$. Hence, we can choose a sequence $z_n \in X$ with limit point $z \in A_C(x)$ such that $d(z_n, y) \geq r$. In the limit we have $d(z, y) \geq r$ and so $d_+(A_C(x), y) \geq r$ for all $y \in X$ and $x \in C$.

In particular, for every $y \in C$, there exists $z \in A_C(x)$ such that $d(y, z) \geq r$ and so $d(y, z) = r$. That is, z is antipodal to y . Thus, $A_C \circ A_C(x) = C$ for all $x \in C$.

□

There is a geometric condition on a metric space which does give us the result we want.

Definition 1.11. *Let X be a metric space with metric d .*

We will say that a point $x \in X$ lies between points $x_0, x_1 \in X$ when for all $y \in X$

$$(1.13) \quad d(x, y) \leq \max(d(x_0, y), d(x_1, y)) \quad ,$$

with strict inequality when $d(x_0, y) \neq d(x_1, y)$.

A semi-geodesic between x_0 and x_1 is a connected subset $G(x_0, x_1)$ of X such that $x_0, x_1 \in G(x_0, x_1)$ and x lies between x_0 and x_1 for all $x \in G(x_0, x_1) \setminus \{x_0, x_1\}$.

We call the metric d semi-geodesic when between any two points of X there exists a semi-geodesic.

If the metric is semi-geodesic then we call $C \subset X$ s -convex when any semi-geodesic between two points of C is contained in C and we will call C w -convex when between any two points of C some semi-geodesic between them is contained in C .

Proposition 1.12. Assume that X has a semi-geodesic metric d .

- (a) An s -convex set is w -convex set and a w -convex set is connected.
- (b) The intersection of any collection of s -convex sets is s -convex.
- (c) If $G(x_0, x_1)$ is a semi-geodesic between distinct points x_0 and x_1 then x_0 and x_1 are contained in the closure of $G(x_0, x_1) \setminus \{x_0, x_1\}$.

- (d) For all $x_0, x_1, x \in X$

$$(1.14) \quad x \text{ between } x_0 \text{ and } x_1 \implies \max(d(x, x_0), d(x, x_1)) \leq d(x_0, x_1).$$

with strict inequality unless $x_0 = x_1$.

Let x lie between x_0 and x_1 . If $x_0 \neq x_1$ then $x \neq x_0$ and $x \neq x_1$. If $x_0 = x_1$ then $x = x_0 = x_1$.

- (e) For $x_0, x_1 \in X$ the set

$$(1.15) \quad \bar{G}(x_0, x_1) =_{\text{def}} \{x_0, x_1\} \cup \{x \text{ lies between } x_0 \text{ and } x_1\}$$

is an s -convex, semi-geodesic between x_0 and x_1 . It is maximal, i.e. it contains every semi-geodesic between x_0 and x_1 . Furthermore, $\bar{G}(x_0, x_1) \setminus \{x_0, x_1\}$ is s -convex.

Proof: (a): Clearly, s -convex implies w -convex. If $x_0 \in C$ is fixed and C is w -convex then we can express C as the union of semi-geodesics $G(x_0, x_1)$ contained in C with x_1 varying over C . Hence, C is connected.

- (b): Obvious.

(c): If x_0 were not in the closure of $G(x_0, x_1) \setminus \{x_0, x_1\}$ then $\{x_0\}$ would be a clopen subset of $G(x_0, x_1)$ and it is a proper subset since it does not contain x_1 . This contradicts connectedness of $G(x_0, x_1)$.

(d): With $y = x_1$ we have $d(x, x_1) = d(x, y) \leq \max(d(x_0, y), d(x_1, y)) = d(x_0, x_1)$. If $d(x_0, x_1) = 0$, then $d(x, x_1) = 0$. If $d(x_0, x_1) > 0$ then the inequality is strict. Similarly, for $y = x_0$.

(e): From (d) we can restrict to the case when $x_0 \neq x_1$. Let $z_0, z_1 \in \bar{G}(x_0, x_1)$. For $y \in X$ we have

$$(1.16) \quad \max(d(y, z_0), d(y, z_1)) \leq \max(d(y, x_0), d(y, x_1))$$

because each of z_0, z_1 is either between x_0 and x_1 or equal to one of them.

Assume that z lies between z_0 and z_1 so that

$$(1.17) \quad d(y, z) \leq \max(d(y, z_0), d(y, z_1))$$

with strict inequality unless $d(y, z_0) = d(y, z_1)$. If $z_0 = z_1$ then by (d) again $z = z_0 = z_1$ and so $z \in G(x_0, x_1)$. Now assume that $z_0 \neq z_1$. We show that z lies between x_0 and x_1 .

From (1.16) and (1.17) we obtain $d(y, z) \leq \max(d(y, x_0), d(y, x_1))$. If $d(y, z) = \max(d(y, x_0), d(y, x_1))$ then equality holds in (1.16) and (1.17) as well. From equality in (1.17) we have $d(y, z_0) = d(y, z_1)$ because z lies between z_0 and z_1 . From equality in (1.16) this common value is $\max(d(y, x_0), d(y, x_1))$. If either z_0 or z_1 lies between x_0 and x_1 then we obtain $d(y, x_0) = d(y, x_1)$. The remaining case is $\{z_0, z_1\} = \{x_0, x_1\}$ so that here $d(y, x_0) = d(y, x_1)$ holds as well because $d(y, z_0) = d(y, z_1)$. As y was arbitrary, z lies between x_0 and x_1 .

This argument shows that both $\bar{G}(x_0, x_1)$ and $\bar{G}(x_0, x_1) \setminus \{x_0, x_1\}$ are s -convex and so they are each connected by (a). Hence, $\bar{G}(x_0, x_1)$ is a semi-geodesic between x_0 and x_1 . Because it is s -convex and contains x_0 and x_1 , it contains any semi-geodesic between them and so is maximal. \square

Lemma 1.13. *Let X be a metric space with a semi-geodesic metric d .*

- (a) *The metric d is connected.*
- (b) *Let $r > 0$. If C is a nonempty r -bounded subset of X then C_r^* is a closed, s -convex set. If C is compact then $\{d_+(C, \cdot) < r\}$ is a nonempty, open, s -convex set with closure C_r^* and if C is w -convex then $C \cap \{d_+(C, \cdot) < r\}$ is nonempty.*

Proof: It is clear from the definitions that the balls $V_r(y)$ and $\bar{V}_r(y)$ are s -convex. $C_r^* = \{d_+(C, \cdot) \leq r\} = \bigcap \{\bar{V}_r(y) : y \in C\}$ is s -convex and if C is compact then $\{d_+(C, \cdot) < r\} = \bigcap \{V_r(y) : y \in C\}$ and so it too is s -convex.

(a): Now assume C is compact and that $d_+(C, x_0) < r$ and $d_+(C, x_1) \leq r$. Let $G(x_0, x_1)$ be a semi-geodesic between x_0 and x_1 . If $x \in G(x_0, x_1) \setminus \{x_0, x_1\}$, and so x lies between x_0 and x_1 , then for all $y \in C$ $d(x_0, y) < r$ and $d(x_1, y) \leq r$ and so $d(x, y) < r$. Hence, $d_+(C, x) < r$. By Proposition 1.12(c) x_1 is in the closure of $G(x_0, x_1) \setminus \{x_0, x_1\}$ and so of $\{d_+(C, \cdot) < r\}$. It follows that the metric is connected.

(b): Now assume that C is compact so that we can write C as the union of compact, nonempty subsets C_1, \dots, C_N each with diameter less than r . Choose $x_i \in C_i$ for $i = 1, \dots, N$. Begin with $y_1 = x_1$. Because C is r -bounded and the diameter of C_1 is less than r , we have $d_+(C_1, y_1) < r$ and $d_+(C, y_1) \leq r$, i.e. $y_1 \in C_r^* \cap C$.

MAXIMAL R-DIAMETER SETS AND SOLIDS OF CONSTANT WIDTH

Assume we have chosen $y_k \in C_r^*$ with $d_+(C_1 \cup \dots \cup C_k, y_k) < r$ for some $k < N$. If $y_k = x_{k+1}$ then let $y_{k+1} = y_k = x_{k+1}$. Otherwise, let $G(y_k, x_{k+1})$ be a semi-geodesic between the two points. It is contained in C_r^* by s-convexity. Furthermore, for z and close enough to y_k is it true that $d_+(C_1 \cup \dots \cup C_k, z) < r$ by continuity. On the other hand, $d_+(C_{k+1}, x_{k+1}) < r$ and $d_+(C_{k+1}, y_k) \leq r$ because $y_k \in C_r^*$. By compactness of C_{k+1} it follows that $d_+(C_{k+1}, z) < r$ for all $z \in G(y_k, x_{k+1}) \setminus \{y_k, x_{k+1}\}$. By Proposition 1.12 (c) again we can choose $y_{k+1} \in G(y_k, x_{k+1}) \setminus \{y_k, x_{k+1}\}$ and close enough to y_k that

$$(1.18) \quad \begin{aligned} d_+(C_1 \cup \dots \cup C_{k+1}, y_{k+1}) &= \\ \max(d_+(C_1 \cup \dots \cup C_k, y_{k+1}), d_+(C_{k+1}, y_{k+1})) &< r. \end{aligned}$$

Furthermore, if $y_k \in C$ and C is w-convex then we can choose the semi-geodesic $G(y_k, x_{k+1})$ to lie in C and so obtain $y_{k+1} \in C$ as well.

It follows by induction that there exists $x \in X$ such that $d_+(C, x) = d_+(C_1 \cup \dots \cup C_N, x) < r$. Hence, the open s-convex set $\{d_+(C, \cdot) < r\}$ is nonempty. Its closure is C_r^* by (a). Finally, if C is w-convex then the construction yields a point of $C \cap \{d_+(C, \cdot) < r\}$.

□

Theorem 1.14. *Let X be a metric space with a proper, semi-geodesic metric. Let C be a closed, r -bounded subset of X with r a regular value for C . The following are equivalent*

- i. C is r -maximal.
- ii. C is s-convex and satisfies the antipodal condition.
- iii. C is w-convex and satisfies the antipodal condition.
- iv. C has a nonempty interior and satisfies the antipodal condition.

If C is r -maximal then it is an s-convex, regular closed subset with interior $\{d_+(C, \cdot) < r\}$.

Proof: (i) \Rightarrow (ii): If C is r -maximal then it is compact because the metric is proper and so by Lemma 1.13(b) $C = C_r^*$ is s-convex. The antipodal condition follows from Theorem 1.6.

(ii) \Rightarrow (iii): Obvious.

(iii) \Rightarrow (iv): Because C is w-convex, Lemma 1.13(b) implies there exists a point $x \in C \cap \{d_+(C, \cdot) < r\}$. That is, $x \in C$ with no antipode in C . By the antipodal condition every boundary point of C has an antipode in C . Hence, x is an interior point of C .

(iv) \Rightarrow (i): By Lemma 1.13(a) the metric is connected. Hence, if C is r -bounded, has a nonempty interior and satisfies the antipodal condition then it is r -maximal by Theorem 1.9(c).

Applying Lemma 1.13(b) again we see that if C is r -maximal then $C = C_r^*$ is the closure of the nonempty open set $\{d_+(C, \cdot) < r\}$. The latter equals the interior by Corollary 1.7(a).

□

2. MINKOWSKI SPACES

Now we assume that E is a Minkowski space, that is, a vector space with finite dimension n , equipped with a norm $|\cdot|$. Let S denote the unit sphere in E , so that $S = A_1(0)$. For $x, y \in E$ we will let $[x, y]$ denote the closed segment $\{tx + (1-t)y : t \in [0, 1]\}$ and similarly we will use (x, y) , $[x, y)$ and $(x, y]$ when either or both of the endpoints are omitted.

We first review some elementary facts about such spaces and their convex subsets.

Proposition 2.1. *Let $|\cdot|_0$ be usual Euclidean norm on \mathbb{R}^n with unit sphere S_0 and let $|\cdot|$ be another norm on \mathbb{R}^n . Let C be a nonempty convex subset of \mathbb{R}^n .*

(a) *There exist real numbers $M \geq m > 0$ such that for all $x \in \mathbb{R}^n$*

$$(2.1) \quad m|x|_0 \leq |x| \leq M|x|_0.$$

In particular, the topologies and concepts of boundedness induced by the norms $|\cdot|_0$ and $|\cdot|$ agree.

(b) *If $x_0 \in C$ and $x_1 \in C^\circ$ then $(x_0, x_1] \subset C^\circ$.*

(c) *If C is closed and bounded and $C^\circ \neq \emptyset$ then there is a homeomorphism h on \mathbb{R}^n which maps the unit ball onto C . C is the closure of its interior and $\text{Bdry}(C)$ is homeomorphic to S .*

(d) *S_0 is homeomorphic to S .*

(e) *If \hat{E} is the affine subspace of E generated by C then the interior of C with respect to \hat{E} is nonempty.*

(f) *The metric on E given by $d(x, y) = |x - y|$ is proper, open, semi-geodesic and connected. For a subset C of E , s -convexity implies convexity implies w -convexity.*

Proof: (a): If $x = (t_1, \dots, t_n)$ then $|x| \leq \sum_{i=1}^n |t_i| |e_i|$ where $\{e_1, \dots, e_n\}$ is the standard basis for \mathbb{R}^n . Hence, then norm $|\cdot|$ is a continuous function on \mathbb{R}^n with respect to the original, Euclidean space topology. It is positive on the compact sphere S_0 and so (2.1) holds with m and M the minimum and maximum values on the norm on S_0 .

(b): By translating if necessary we can assume $x_0 = 0$. If $V_\epsilon(x_1) \in C$ then $V_{t\epsilon}(tx_1) = tV_\epsilon(x_1) \subset C$.

(c): By translating we can assume that $0 \in C^\circ$. Then $r(x) = x/|x|$ defines a continuous map from $Bdry(C)$ to S . If $u \in S$ then $tu \in C^\circ$ for t close to 0. Since C is closed and bounded, and hence compact, there exists $0 < M(u) < \infty$ such that $M(u) = \max\{t : tu \in C\}$. By (b) $tu \in C^\circ$ for all $0 \leq t < M(u)$. Hence, $x = M(u)u$ is the unique point of the boundary of C such that $r(x) = u$. As r is a continuous bijection between compacta, it is a homeomorphism. Hence, $u \mapsto M(u) = |r^{-1}(u)|$ is a continuous, positive real-valued function on S . Let $h(x) =_{def} M(x/|x|)x$ for $x \neq 0$ and $h(0) = 0$ with $h^{-1}(y) = M(y/|y|)^{-1}y$ for $y \neq 0$. Because M and M^{-1} are bounded, h and its inverse are continuous at 0. Clearly, h maps the unit ball to C .

(d): Since the sphere is the boundary of the unit ball, this follows from (c).

(e): Assume that $\{x_0, \dots, x_m\}$ is a maximal affinely independent subset of C . Again we translate to assume $x_0 = 0$ so the $\{x_1, \dots, x_m\}$ is a basis for the linear subspace A generated by C . The open simplex $\{\sum_{i=1}^m t_i x_i : \text{with } 0 < t_1, \dots, t_m \text{ and } \sum_{i=1}^m t_i < 1\}$ is a nonempty subset of C which is open in \hat{E} (by (a) applied to \hat{E}).

(f): The metric is obviously open. It is proper because E is finite dimensional. For $x_0, x_1 \in E$ the linear path $\{(1-t)x_0 + tx_1 : t \in [0, 1]\}$ is a semi-geodesic because $|x_t - y| \leq (1-t)|x_0 - y| + t|x_1 - y|$ for $t \in [0, 1]$. The metric is connected by Lemma 1.13(a). Since a segment between two points is a semi-geodesic it is clear that the ordinary concept of convexity lies between the concepts of s-convexity and w-convexity.

□

Remark: By choosing a basis we can export the results to any normed linear space of dimension n .

Example: Let $E = \mathbb{R}^n$ with $n > 1$ equipped with the ℓ_∞ norm, $|(a_1, \dots, a_n)| = \max(|a_1|, \dots, |a_n|)$. If $x_0 = (a_1, \dots, a_n)$ and $x_1 = (b_1, \dots, b_n)$ with $a_i < b_i$ for $i = 1, \dots, n$ then $x = (c_1, \dots, c_n)$ lies between x_0 and x_1 iff $a_i < c_i < b_i$ for $i = 1, \dots, n$. Adjoining $\{x_0, x_1\}$ to the product of the open intervals $(a_1, b_1), \dots, (a_n, b_n)$ we obtain the s-convex set $\bar{G}(x_0, x_1)$ in E , see Proposition 1.12 (e). Hence, the segment $[x_0, x_1]$ is convex but not s-convex. If x_t is a continuous path from x_0 to x_1 which is increasing in each coordinate then the image is a w-convex set but, unless it is the segment, it is not convex.

For the rest of this section we assume that E is a Minkowski space.

Theorem 2.2. *Let $C \subset E$ be an r -bounded closed set.*

- (a) $C_r^* = \{d_+(C, \cdot) \leq r\}$ *is a compact, convex set with a nonempty interior equal to $\{d_+(C, \cdot) < r\}$.*
 - (b) *The following are equivalent*
 - i. C *is r -maximal.*
 - ii. C *is convex and satisfies the antipodal condition.*
 - iii. C *has a nonempty interior and satisfies the antipodal condition.*
- If C is r -maximal then it is a convex, regular closed subset with interior $\{d_+(C, \cdot) < r\}$.*

Proof: Part (a) follows from Lemma 1.13 and part (b) follows from Theorem 1.14.

□

Now let E^* denote the dual space of E , the space of real linear functionals on E . The space E^* is equipped with the norm $|\omega| = \max\{\omega(x) : x \in S\}$. We can use the max instead of the sup because S is compact and the linear functionals are continuous. Thus, for all $(\omega, x) \in E^* \times E$, $|\omega(x)| \leq |\omega| \cdot |x|$. Let S^* denote the unit sphere in E^* .

For any compact convex subset C of E let

$$(2.2) \quad G_C^* =_{\text{def}} \{(\omega, x) \in S^* \times \text{Bdry}(C) : \omega(x) = \max\{\omega(y) : y \in C\}\},$$

which we will call the *Gauss relation*. For the special case of the unit ball, we define

$$(2.3) \quad G_1^* =_{\text{def}} \{(\omega, x) \in S^* \times S : \omega(x) = 1\}.$$

Proposition 2.3. *Let E be a finite dimensional normed linear space with dual space E^* . The relation $G_1^* \subset S^* \times S$ is a closed, surjective relation. If C is a compact, convex set then $G_C^* \subset S^* \times \text{Bdry}(C)$ is a closed, surjective relation. If C is the unit ball in E then $G_C^* = G_1^*$.*

Proof: The relation G_1^* is obviously closed. For each $y \in C$ the set $\{(\omega, x) : \omega(x) \geq \omega(y)\}$ is obviously closed. Intersecting over all $y \in C$ we obtain the closed relation G_C^* .

Fix $\omega \in S^*$. By compactness there exists $x \in C$ at which ω achieves its maximum value. Since a nonzero linear functional is an open map, this maximum point cannot occur in the interior. If C is the unit ball

then since $|\omega| = 1$ this maximum is 1 by definition of the norm on E^* . Hence, $G_C^* = G_1^*$ when C is the unit ball. Since ω was arbitrary $(G_C^*)^{-1}(S) = S^*$.

Now fix $x \in \text{Bdry}(C)$. There exists $\omega \in S^*$ whose kernel is the translate of a hyperplane of support through x for the compact convex set C . If $\omega(x)$ is not the maximum on C then it is the minimum and so $-\omega \in S^*$ takes its maximum at x . Hence, $G_C^*(S^*) = \text{Bdry}(C)$.

□

Corollary 2.4. *For $x_1, x_2 \in E$, $|x_1 - x_2| = \max\{\omega(x_1 - x_2) : \omega \in S^*\}$.*

Proof: Since $|\omega| = 1$, $\omega(x_1 - x_2) \leq |x_1 - x_2|$ with equality if $x_1 = x_2$. When the two points are distinct define $x = (x_1 - x_2)/|x_1 - x_2| \in S$ and let $\omega \in (G_1^*)^{-1}(x)$ to get equality.

□

For any compact subset $C \subset E$ and $\omega \in S^*$ we define the ω diameter of C to be

$$\begin{aligned} \text{diam}_\omega(C) &= \max\{\omega(x_1 - x_2) : x_1, x_2 \in C\} \\ (2.4) \quad &= \max\{\omega(x_1) : x_1 \in C\} - \min\{\omega(x_2) : x_2 \in C\}. \end{aligned}$$

From Corollary 2.4 it follows that

$$(2.5) \quad \text{diam}(C) = \sup_{\omega \in S^*} \text{diam}_\omega(C).$$

Theorem 2.5. *Let C be an r -bounded subset of E .*

(a) *For $x_1, x_2 \in C$ and $\omega \in S^*$*

$$(2.6) \quad \omega(x_1) - \omega(x_2) = r.$$

implies that the pair x_1, x_2 is antipodal, $\text{diam}_\omega(C) = r$ and

$$(2.7) \quad \begin{aligned} \omega(x_1) &= \max\{\omega(x) : x \in C\}, & \text{and} \\ \omega(x_2) &= \min\{\omega(x) : x \in C\}. \end{aligned}$$

(b) *If $x_1, x_2 \in C$ is an antipodal pair, i.e. $|x_1 - x_2| = r$, then there exists $\omega \in S^*$ such that $\omega(x_1) - \omega(x_2) = r$.*

(c) *If $\omega \in S^*$ with $\text{diam}_\omega(C) = r$ then there exists an antipodal pair $x_1, x_2 \in C$ such that $\omega(x_1) - \omega(x_2) = r$.*

Proof: If $|x_1 - x_2| = r$ then by Corollary 2.4 there exists $\omega \in S^*$ such that equation (2.6) holds. Conversely, if there exists such an $\omega \in S^*$ then by Corollary 2.4 $|x_1 - x_2| \geq r$. Equality follows because C is r -bounded and so x_1 and x_2 are antipodal. Clearly, (2.6)

implies $\text{diam}_\omega(C) \geq r$ and so equality holds because $r \geq \text{diam}(C) \geq \text{diam}_\omega(C)$.

If $\text{diam}_\omega(C) = r$ then there exists a pair $x_1, x_2 \in C$ such that (2.6) holds.

Now if equation (2.6) holds and $y \in C$ with $\omega(y) \geq \omega(x_1)$ then since C is r -bounded,

$$(2.8) \quad r \geq |y - x_2| \geq \omega(y) - \omega(x_2) \geq \omega(x_1) - \omega(x_2) = r.$$

Hence, $\omega(y) = \omega(x_1)$ and so $\omega(x_1)$ is the maximum. Similarly, $\omega(x_2)$ is the minimum.

□

Definition 2.6. A subset C has constant diameter if

$$(2.9) \quad \text{diam}(C) = \text{diam}_\omega(C) \quad \text{for all} \quad \omega \in S^*.$$

Theorem 2.7. Assume that a closed, convex set C has diameter r .

- (a) C has constant diameter iff for every $\omega \in S^*$ there exist $x_1, x_2 \in C$ such that $\omega(x_1) - \omega(x_2) = r$.
- (b) If C has constant diameter then C is r -maximal.

Proof: (a): Apply Theorem 2.5.

(b): If $x_1 \in \text{Bdry}(C)$ and $\omega \in (G_C^*)^{-1}(x_1)$ then by definition $\omega(x_1)$ is the maximum of ω on C . Let x_2 be the point of C where ω has its minimum. Then $\omega(x_1) - \omega(x_2) = \text{diam}_\omega(C) = r$. Hence, by Theorem 2.5 again $|x_1 - x_2| = r$ and so x_1 and x_2 are antipodal. As every point of the boundary of C has an antipode, C is r -maximal by Theorem 2.2.

□

For any closed, convex r -bounded subset C of E let

$$(2.10) \quad K_C^* =_{\text{def}} \{ (\omega, x) \in S^* \times \text{Bdry}(C) : \text{for some } x_2 \in C \quad \omega(x) - \omega(x_2) = r \}.$$

Theorem 2.8. Let C be a closed, convex r -bounded subset of E .

- (a) K_C^* is a closed subset of G_C^* .
- (b) C is r -maximal iff $K_C^*(S^*) = \text{Bdry}(C)$, ie. K_C^* projects onto the second coordinate.
- (c) The following conditions are equivalent:
 - (i) $(K_C^*)^{-1}(\text{Bdry}(C)) = S^*$.
 - (ii) K_C^* is a surjective relation from S^* to $\text{Bdry}(C)$.
 - (iii) $K_C^* = G_C^*$.

(iv) C has constant diameter.

Proof: (a): It is easy to check that K_C^* is closed. From Theorem 2.5 it follows that K_C^* is contained in G_C^* , ie. $\omega(x) - \omega(x_2) = r$ implies $\omega(x)$ is the maximum of ω on C .

(b): By Theorem 2.2 C is r -maximal iff for every $x \in \text{Bdry}(C)$ there exists $x_2 \in C$ such that $|x - x_2| = r$ and by Corollary 2.4 this is true iff $\omega(x) - \omega(x_2) = r$ for some $\omega \in S^*$.

(c): Since G_C^* is a surjective relation by Proposition 2.3 it is clear that (iii) \Rightarrow (ii) \Rightarrow (i). Now assume (i) and let $(\omega, x) \in G_C^*$. By (i) there exist x_1, x_2 such that $\omega(x_1) - \omega(x_2) = r$. By (a) $(\omega, x_1) \in G_C^*$ and so the maximum of ω on C is $\omega(x_1) = \omega(x)$. Hence, $\omega(x) - \omega(x_2) = r$ and so $(\omega, x) \in K_C^*$. Thus, (i), (ii) and (iii) are equivalent.

By Theorem 2.7(a) (iv) is equivalent to (i).

□

Remark: Notice that if $\omega(x_1) - \omega(x_2) = r$ then $(\omega, x_1) \in K_C^*$ and $(-\omega, x_2) \in K_C^*$.

Example: When C is the unit ball, $G_C^* = G_1^*$ by Proposition 2.3. It follows that for every $\omega \in S^*$ the maximum value on S is 1 and the minimum value is -1 . Hence, the unit ball has constant diameter 2. It follows that any ball in E of radius $r/2$ has constant diameter r and is r -maximal.

If $n = 1$ then E is isometric to \mathbb{R} . In a one-dimensional normed linear space the closed balls of radius $r/2$, i.e. the closed intervals of length r , are all of the r -maximal sets.

For any n if E is \mathbb{R}^n equipped with the ℓ_∞ norm: $|(x_1, \dots, x_n)| = \max(|x_1|, \dots, |x_n|)$ then the closed balls of radius $r/2$ are all of the r -maximal sets. To see this, project the r -maximal set C to coordinate i . The image is contained in some interval I_i of length r . The product of the intervals $I_1 \times \dots \times I_n$ is a ball of radius $r/2$ and so is r -maximal. The product contains C and so, by maximality, equals C .

C is called *strictly convex* if whenever x_0, x_1 are two distinct elements of C then $(x_0, x_1) \subset C^\circ$. A nonempty, strictly convex set is either a singleton or else has a nonempty interior.

Proposition 2.9. *If C is a strictly convex, closed, bounded set then every $\omega \in S^*$ achieves its maximum value on C at a unique point which lies in $\text{Bdry}(C)$. The relation G_C^* is a continuous function from S^* onto $\text{Bdry}(C)$.*

Proof: If $a = \omega(x_0) = \omega(x_1)$ with $x_0 \neq x_1$ in C then $a = \omega(x_t)$ with $x_t = tx_1 + (1-t)x_0$ for all $t \in [0, 1]$. By strict convexity, $x_t \in C^\circ$ for $t \in (0, 1)$. Because ω is an open map to \mathbb{R} the value a cannot be a maximum.

Hence, the closed relation G_C^* is a function from S^* to $Bdry(C)$. It is easy to check that a function which is closed relation between compacta is continuous, see Akin (1992) Corollary 1.2.

□

Corollary 2.10. *If C is a strictly convex, closed, r -bounded subset then K_C^* is a continuous function on a closed subset $dom(K_C^*) \subset S^*$. If $\omega \in dom(K_C^*)$ then $-\omega \in dom(K_C^*)$.*

C is r -maximal iff K_C^ is a surjective map from $dom(K_C^*)$ to $Bdry(C)$ and C has constant diameter iff $dom(K_C^*) = S^*$.*

Proof: The statements in the first paragraph are immediate from Proposition 2.9 and the Remark following Theorem 2.8. The domain of the function K_C^* is the closed subset $dom(K_C^*) = (K_C^*)^{-1}(Bdry(C))$.

The second paragraph follows from Theorem 2.8 itself.

□

The norm $|\cdot|$ on E is called *strictly convex* when the unit ball $V_1(0)$ is strictly convex. Since we obtain any other ball by dilating and translating it then follows that every closed ball is strictly convex.

Theorem 2.11. *Assume that E has a strictly convex norm. If C is a compact subset of E and $r > 0$ then $C_r^* = \{d_+(C, \cdot) \leq r\}$ is strictly convex. In particular, if C is r -maximal then it is strictly convex.*

Proof: If for distinct points x_1, x_2 $d_+(C, x_1), d_+(C, x_2) \leq r$ then for every $y \in C$ we have $x_t \in V_r(y)$ for each $x_t \in (x_1, x_2)$. By compactness $d(y, x_t) < r$ for all $y \in C$ implies $d_+(C, x_t) < r$ and so $x_t \in \{d_+(C, \cdot) < r\}$ which is the interior of C_r^* .

□

We conclude this section with some results we will use later.

Proposition 2.12. *Assume that C is a bounded closed set in E with a nonempty interior. If for every point $x \in Bdry(C)$ there exists a hyperplane of support for C containing x then C is convex.*

Proof: Let $U = C^\circ$ so that U is nonempty. The assumption about hyperplanes of support says that for every $x \in Bdry(C)$ there exists

$\omega_x \in S^*$ which achieves its maximum on C at x . Let

$$(2.11) \quad \hat{C} =_{def} \bigcap_{x \in Bdry(C)} \{y \in E : \omega_x(y) \leq \omega_x(x)\}.$$

\hat{C} is a closed, convex set which contains C and so has nonempty interior. Recall that each $\omega \in S^*$ is an open map. Hence, for each $x \in Bdry(C)$ there exist $z \in E$ arbitrarily close to x with $\omega_x(z) > \omega_x(x)$ and so with $z \notin \hat{C}$. Thus, $Bdry(C) \subset Bdry(\hat{C})$. Since the boundary of \hat{C} is disjoint from its interior it follows that U is a clopen subset of the interior of \hat{C} . By Proposition 2.1 (c) the interior of \hat{C} is homeomorphic to the open unit ball and so is connected. Hence, U is the interior of \hat{C} . Because \hat{C} is the closure of its interior, we have

$$(2.12) \quad \hat{C} = \overline{U} \subset C = U \cup Bdry(C) \subset \hat{C}.$$

Thus, $C = \hat{C}$.

□

Theorem 2.13. *Let $\{C_n\}$ be a sequence of compact subsets of E which converge to C in the Hausdorff metric. If each C_n is r -maximal then C is r -maximal. If each C_n is a convex set with constant diameter r then C is a convex set with constant diameter r .*

Proof: Just as with (1.6) it is easy to check that for any $\omega \in S^*$ and compacta C, D :

$$(2.13) \quad \begin{aligned} |diam(C) - diam(D)| &\leq 2d(C, D), \\ |diam_\omega(C) - diam_\omega(D)| &\leq 2d(C, D) \end{aligned}$$

Hence, the conditions $diam(C) \leq r$, $diam(C) = r$ and $diam_\omega(C) = r$ are preserved by limits in the Hausdorff metric.

The function $q : E \times E \times [0, 1] \rightarrow E$ by $(x_0, x_1, t) \mapsto (1-t)x_0 + tx_1$ is continuous and so the association $C \mapsto q(C \times C \times [0, 1])$ is continuous with respect to the Hausdorff metric (see, e.g., Akin (1993) Proposition 7.16). Hence, the set of $\{C : C = q(C \times C \times [0, 1])\}$ is a closed set. This is the collection of convex sets.

It follows that if each C_n is a convex set of constant diameter r then so is C .

Now if each C_n is r -maximal, then each is convex and so C is convex. By Theorem 1.6 C satisfies the antipodal condition. By Theorem 2.2 (c) C is r -maximal.

□

It is a classic result, which we will reprove in the next section, that for a Euclidean an r -bounded subset is r -maximal iff it has constant width. That is, in the Euclidean case the converse of Theorem 2.7 (b) holds. Eggleston (1965) gave examples of Minkowski spaces with r -maximal subsets which are not of constant width.

3. EUCLIDEAN SPACES

From now on we assume that E is a Euclidean space of dimension $n \geq 2$. That is, E is an n dimensional linear space with norm given by an inner product and so isometric to \mathbb{R}^n with the usual metric. If x_1, x_2 are distinct points of norm 1 then

$$\begin{aligned} (3.1) \quad |tx_1 + (1-t)x_2|^2 &= t^2|x_1|^2 + 2t(1-t)x_1 \cdot x_2 + (1-t)^2|x_2|^2 \\ &= 1 - t(1-t)(1 - \cos \theta) \end{aligned}$$

which is less than 1 since the angle θ between x_1 and x_2 is not 0. That is, the norm is strictly convex. It follows from Theorem 2.11 that for $C \subset E$ compact and $r > 0$ the dual C_r^* is strictly convex. However, in this case we can do better.

Let y_1 and y_2 be two distinct points of the Euclidean space E . For $x \in X$ with $d(x, y_1) = d(x, y_2) = r$ there is a unique circle in X which is centered at x and which passes through y_1 and y_2 . The *arc α between y_1 and y_2 with center x* is the smaller of the two arcs of this circle with endpoints y_1 and y_2 . If x is the midpoint of $[y_1, y_2]$ then either semicircle with endpoints y_1 and y_2 is considered to be “the” arc α . The arc α is said to have radius the common distance r . We call $S \setminus \{y_1, y_2\}$ the *open arc between y_1 and y_2 with center x* .

Definition 3.1. *Let E be a Euclidean space of dimension at least 2 and let $r > 0$. A subset C of X is called r -convex if it is convex and $\alpha \subset C$ whenever α is an arc of radius at least r between two points of C .*

Notice that as r tends to infinity the arcs between y_1 and y_2 tend to the segment $[y_1, y_2]$. From this it is not hard to show that convexity actually follows from the arc condition. Clearly the closure of an r -convex set is r -convex and any intersection of r -convex sets is r -convex. For $C \subset E$ the r -convex hull of C is the intersection of all r -convex sets which contain C or, equivalently, the minimum r -convex set containing C .

Proposition 3.2. *If C is r -convex then it is strictly convex.*

Proof: For y_1, y_2 two distinct points of C with $d(y_1, y_2) = 2\epsilon$ let y be the midpoint of the segment between them and P be the hyperplane through y perpendicular to the line through them. Choose $a > \max(r, \epsilon)$. Let S be the sphere in P centered at y with radius $a - \sqrt{a^2 - \epsilon^2}$. For each point of S lies on an arc between y_1 and y_2 of radius $a > r$. So by r -convexity $S \subset C$. By convexity C contains the convex hull of $S \cup \{y_1, y_2\}$ whose interior contains every point on the open segment between y_1 and y_2 . Thus, C is strictly convex. \square

We now require a bit of plane geometry.

Lemma 3.3. *Assume P is a Euclidean plane. Let α be an arc between distinct points $y_1, y_2 \in X$ with center x and radius r . Let B be the ball of radius r centered at x so that $\alpha \subset S$, the boundary circle of B . Let B_0 be a ball of radius $r_0 > 0$ and boundary circle S_0 . If $\alpha \subset B_0$ and $\alpha \cap S_0$ contains a point of the open arc $\alpha \setminus \{y_1, y_2\}$ then $r_0 \geq r$ and if $r_0 = r$ then x is the center of B_0 and so $S_0 = S \supset \alpha$.*

Proof: Let $y \in S_0 \cap \alpha \setminus \{y_1, y_2\}$. If α intersects S_0 transversely at y then part of α near y on one side or the other lies outside of B_0 contra assumption. Hence, α intersects S_0 tangentially at y . The center x of α lies on the line through y of the perpendicular to the tangent of α at y . The center x_0 of S_0 must lie on the same line. If x and x_0 lay on opposite sides of y on the line, then $B \cap B_0 = \{y\}$ again contradicting $\alpha \subset B_0$. If x_0 is on the open segment between x and y (i.e. $r_0 < r$) then $B_0 \subset B$ with $B_0 \cap S = \{y\}$ and so $B_0 \cap \alpha = \{y\}$ which is still false. Hence, $r_0 \geq r$. Furthermore, if $r_0 = r$ then $x_0 = x$, $B_0 = B$ and $S_0 = S$. \square

Lemma 3.4. *A closed ball B in E of radius $r > 0$ is r -convex.*

Proof: Since B closed and the arcs of radius a move continuously with the endpoints it suffices to consider the case when y_1, y_2 are in the interior of B . For any arc between y_1 and y_2 the three points y_1, y_2 and the center x determine a plane P in E . Any plane through y_1 and y_2 intersects B in a disc B_1 with radius $r_1 \leq r$. Let x move along the line L in P which is the perpendicular bisector of the segment $[y_1, y_2]$ and for each such x let α be the arc between y_1 and y_2 centered at x .

As x moves toward infinity the arc α approaches the segment $[y_1, y_2]$ which is contained in the interior of B_1 with respect to the plane P .

As x moves from infinity toward the mid-point either all of the arcs remain in the interior of B_1 or there is a first position x^* such that the arc touches the boundary circle of B_1 . This arc in P centered at x^* is contained in B_1 and by Lemma 3.3 has radius less than r_1 and so less than r .

Hence, as long as $d(x, y_1) = d(x, y_2) \geq r$ the arc is contained in the interior of B_1 .

□

From the proof we obtain:

Proposition 3.5. *For two distinct points x_0 and x_1 in a Euclidean space E the only points of E which lie between x_0 and x_1 in the metric space sense are those of the open segment (x_0, x_1) and so $[x_0, x_1]$ is the unique semi-geodesic between x_0 and x_1 . In particular, for a Euclidean space the concepts of s -convexity, convexity and w -convexity agree.*

Proof: To show that no point x of $E \setminus (x_0, x_1)$ lies between x_0 and x_1 it suffices to restrict to the plane through x_0, x_1 and x and so to reduce to dimension two.

Let L be the line through x_0 and x_1 and let \bar{L} be the perpendicular bisector of $[x_0, x_1]$. For any $y \in \bar{L}$ the in-between point x must lie in the disc centered at $\bar{V}_r(y)$ with $r = |x_0 - y| = |x_1 - y|$. As y moves out toward infinity in one direction, the intersections of the discs with the closed half-space on the other side decrease with intersection $[x_0, x_1]$. So if x is in the closed half-space then it must lie in $[x_0, x_1]$.

It follows that $[x_0, x_1]$ is the maximal semi-geodesic $\bar{G}(x_0, x_1)$. On the other hand, removing any points other than x_0 and x_1 disconnects the interval and so no proper subset of $[x_0, x_1]$ is a semi-geodesic between x_0 and x_1 .

The convexity results are then obvious.

□

Proposition 3.6. *For C a bounded subset of E , let D be the r -convex hull of C . The set C_r^* is r -convex and $C_r^* = D_r^*$.*

Proof: By Lemma 3.4 the ball $\bar{V}_r(y)$ is r -convex and so it contains D iff it contains C . Hence, $C_r^* = D_r^*$. Furthermore, C_r^* is the intersection of $\{\bar{V}_r(x) : x \in C\}$ and so is r convex.

□

Theorem 3.7. *If C is an r -maximal set in E , then C is r -convex.*

Proof: By Theorem 1.3(f) $C = C_r^*$. So C is r -convex by Proposition 3.6

□

Corollary 3.8. *Let C be an r -maximal set in E . Assume that for $x \in C$ there are two distinct points $y_1, y_2 \in C$ antipodal to x . Let α the arc between y_1 and y_2 with center x . The angle subtended by α is at most $\pi/3$ and every point of α is a point of $\text{Bdry}(C)$ antipodal to x .*

Proof: The distance $d(y_1, y_2) \leq r$ which equals the radius of the arc. Since the chord between the end-points has length at most the radius, the angle subtended is at most $\pi/3$.

By Theorem 3.7 $\alpha \subset C$. Since x is the center of α , $d(y, x) = r$ for every $y \in \alpha$ and so every point of α is antipodal to x . By Lemma 1.1 $\alpha \subset \text{Bdry}(C)$.

□

Theorem 3.9. *Let C be an r -maximal subset in E . Assume that y_1, y_2 are distinct points of E and that α is an arc between them with radius r and center x . If $\alpha \subset \text{Bdry}(C)$ then $x \in C$ and for each y in the open arc $\alpha \setminus \{y_1, y_2\}$, $A_C(y) = \{x\}$. That is, x is the unique point of C antipodal to y .*

Proof: Let $z \in \text{Bdry}(C)$ be a point antipodal to y . We will show that $z = x$.

Let P be the plane which contains y_1, y_2 and x and let \hat{E} be smallest affine subspace which contains P and z . Let B be the ball in \hat{E} with center z and radius r . Because $\alpha \cup \{z\} \subset C$ and $\text{diam}(C) = r$ we have $\alpha \subset B$. Let S be the boundary sphere of B in \hat{E} .

First we show that $z \in P$ and so $\hat{E} = P$. If not then the intersection $P \cap B$ is a disk in the plane P with radius less than r and with boundary circle $P \cap S$. Furthermore, $\alpha \subset P \cap B$ and $\alpha \cap (P \cap S)$ contains the point y of the open arc. By Lemma 3.3 this can't happen.

Hence, $z \in P$ and so B is a disk in P with radius r which contains α and whose boundary circle meets the point y of the open arc. By Lemma 3.3 again the center z of B agrees with the center x of α . That is, $z = x$ as required.

□

Example: With P the Euclidean Plane let C_0 be a regular polygon with $2k + 1$ vertices ($k \geq 1$) indexed, in order, by the additive group $\mathbb{Z}/(2k + 1)\mathbb{Z}$. Let $r = d(v_i, v_{i+k}) = d(v_i, v_{i-k})$. For other pairs of

vertices, $d(v_i, v_j) < r$. If C_1 is an r -maximal set containing these vertices (and hence by convexity the polygon C_0 and its interior) then C_1 contains the arc between v_{i+n} and v_{i-n} centered at v_i which consists of points antipodal to v_i . The convex hull C of these arcs is a closed subset such that every point of the boundary has antipodal points. Hence, by Theorem 2.2(c) C is r -maximal and contains the vertices. By maximality $C = C_1$. Thus, C is the unique r -maximal set which contains the vertices. These are the Relleaux polyhedrons. With $k = 1$ it is the classical Relleaux triangle.

The vertices need not be evenly spaced. Beginning with the regular polygon with $k \geq 2$ define the $2k + 1$ pointed star to be the graph with vertex v_i connected to v_{i+k} and to v_{i-k} by segments of length r . The star is not rigid and so we can vary the vertices slightly and still have a set of diameter r . Connecting v_{i+k} and v_{i-k} in the new positions by the arc centered at the new position for v_i we get the unique r -maximal set containing the perturbed vertices.

Now we show that in the Euclidean case the r -maximal sets are exactly the sets of constant diameter r , see, e.g. Bonnesen and Fenchel (1987) and Eggleston (1958). For any Minkowski space Theorem 2.7 says that a set of constant diameter r is r -maximal. We prove the converse in the Euclidean case.

In the Euclidean case we can replace the use of the dual space E^* by using E instead. The inner product provides the Riesz Representation isometry from E to E^* by $v \mapsto \omega_v$ where $\omega_v(x) =_{\text{def}} v \cdot x$. On the unit spheres the isomorphism restricts to:

$$(3.2) \quad G_1^* = \{(\omega_u, u) : u \in S\}$$

because for unit vectors u, u_1 we have $u \cdot u_1 = 1$ iff $u_1 = u$. Using this we define for C a compact convex set the relation G_C dual to G_C^* :

$$(3.3) \quad G_C = \{(u, x) \in S \times C : u \cdot x = \max\{u \cdot y : y \in C\}\}.$$

Proposition 3.10. *For a compact convex set C the relation G_C is a closed surjective relation from S to $\text{Bdry}(C)$. If C is strictly convex then G_C is a continuous surjective function from S to $\text{Bdry}(C)$.*

Proof: Since the association $u \mapsto \omega_u$ is a homeomorphism from S to S^* the first result follows from Proposition 2.3 and the second from Proposition 2.9.

□

Thus, when C is a compact strictly convex set, e.g. when C is r -maximal, we can write $x = G_C(u)$ for the unique $x \in \text{Bdry}(C)$ such

that $(u, x) \in G_C$. That is, given $u \in S$, $x = G_C(u)$ is the unique point of C at which $z \mapsto \omega_u(z) = u \cdot z$ takes its maximum on C . Of course, $G_C(u) \in \text{Bdry}(C)$. Following Lachand-Robert and Oudet we will call this function the *inverse Gauss map*.

Now we need another bit of plane geometry.

Lemma 3.11. *Let x_1, x_2 be distinct points of a Euclidean plane P with $|x_1 - x_2| = r$. Let L_1, L_2 be the lines through x_1 and x_2 respectively which are perpendicular to the segment $[x_1, x_2]$ and so are parallel. Let B_1, B_2 be the discs of radius r centered at x_1 and x_2 respectively, with boundary circles S_1, S_2 . Let U be the open strip consisting of the points of P between the lines L_1 and L_2 .*

Assume that $y \in B_1 \cap U$ and that α is the arc between y and x_1 of radius r with center on $S_1 \cap U$. If $\alpha \subset \overline{U}$ then $y \in B_2$.

Proof: Choose coordinates so that x_2 is the origin and $x_1 = (0, r)$. Assume that $y = (a, b)$ with $0 < b < r$. If $a = 0$ then $y \in [x_1, x_2] \subset B_2$. Now assume that $y \notin B_2$.

Without loss of generality we can assume that $a > 0$. Moving along the segment $[y, x_1]$ from y to x_1 let x_3 be the first entrance into B_2 . Thus, $[x_3, x_1]$ is a chord of the circle S_2 and its perpendicular bisector L_3 passes through the center x_2 . The perpendicular bisector of $[y, x_1]$ is parallel to L_3 but to the right of it and so it intersects the semi-circle $S_1 \cap U$ at a point (c, d) with $c > 0$. Then the arc α which connects y and x_1 with center $x_4 = (c, d)$ is tangent at x_1 to the line perpendicular to $[x_4, x_1]$ and is above the tangent line L_1 to S_1 near x_1 . In particular, α is not contained in \overline{U} . Contrapositively, $\alpha \subset \overline{U}$ implies $y \in B_2$.

□

Theorem 3.12. *Assume that C is an r -bounded subset of E ,*

(a) If x_1, x_2 is an antipodal pair in C i.e. $|x_1 - x_2| = r$, and $u = (x_1 - x_2)/r \in S$ then $\omega_u \in S^$ with $\omega_u(x_1) - \omega_u(x_2) = r$ and $\omega_u(x_1) = \max\{\omega_u(y) : y \in C\}$. The hyperplane through x which is perpendicular to u is a hyperplane of support for C which contains x . In particular, if x has more than one antipodal point in C then C has more than one hyperplane of support through x . If C is convex then $(u, x_1) \in G_C$.*

(b) Assume that C is an r -maximal subset of E . If $u \in S$ and $x_1 = G_C(u)$ then $x_2 = x_1 - ru = G_C(-u)$. The pair $\{x_1, x_2\}$ is the unique antipodal pair in C such that the segment $[x_1, x_2]$ is parallel to u .

Proof: (a): Direct computation shows $\omega_u(x_1) - \omega_u(x_2) = r$. The rest follows from Theorem 2.5 and the definition of G_C in the convex case.

(b): This is much more delicate. The main issue is to show that $x_2 \in C$. It will suffice to show that $C \subset \bar{V}_r(x_2)$ and so $x_2 \in C_r^*$, because $C = C_r^*$ by Theorem 1.3 (f).

Now let $y \in C$. If $y = x_1$ or x_2 then $y \in \bar{V}_r(x_2)$. Now assume that y is distinct from x_1 and x_2 and let P be the plane in E which contains x_1, x_2, y . We apply the lemma.

Let $P \cap V_r(x_i)$ be the disc B_i centered at x_i with radius r for $i = 1, 2$. The strip $U = \{w \in P : \omega_u(x_2) < \omega_u(w) < \omega_u(x_1)\}$. Because $x_1 \in C$ and C is r -bounded, $C \subset V_r(x_1)$ and so $P \cap C \subset B_1$. Also for $w \in C$ $\omega_u(w) \leq \omega_u(x_1)$ with equality only with $w = x_1$ by strict convexity of C . Hence, $P \cap C = \bar{U} \cap C \subset \{x_1, x_2\} \cup U \cap C$. Hence, $y \in U \cap C \subset U \cap B_1$. Let α be the arc of radius r which connects x_1 and y with center on a point of $Q_1 \cap U$. Because C is r -convex it follows that the arc $\alpha \subset C \cap P \subset \bar{U}$. Then Lemma 3.11 implies that $y \in B_2$ and so $y \in \bar{V}_r(x_2)$ as required.

Clearly, $r = |x_1 - x_2| = \omega_u(x_1) - \omega_u(x_2)$ so that $\{x_1, x_2\}$ is an antipodal pair with $[x_1, x_2]$ parallel to u . Furthermore, by Theorem 2.5 $u \cdot x_2 = \min\{u \cdot y : y \in C\}$ and so $x_2 = G_C(-u)$. If $[y_1, y_2]$ is parallel to u then $(y_1 - y_2)/r = \pm u$ and by renumbering if necessary we can assume that the sign is positive so that $y_2 = y_1 - ru$. Since $\omega_u(y_1) - \omega_u(y_2) = r$, Theorem 2.5 again implies $\omega_u(y_1)$ is the maximum value of ω_u on C . But x_1 is the unique point $G_C(u)$ where ω_u takes its maximum. Thus, $y_1 = x_1$ and $y_2 = x_2$.

□

Corollary 3.13. *If C is an r -maximal subset of the Euclidean space E then C has constant diameter r . Furthermore, C is the union of the antipodal segments in C . That is, if $x \in C$ then there exists an antipodal pair x_1, x_2 in $\text{Bdry}(C)$ such that $x \in [x_1, x_2]$.*

Proof: Given $\omega \in S^*$ let u be the unit vector of E such that $\omega = \omega_u$. With $x_1 = G_C(u)$ and $x_2 = x_1 - ru$ Theorem 3.12 (b) says that x_1, x_2 is an antipodal pair in C with $\omega(x_1) - \omega(x_2) = r$. Since $\omega \in S^*$ was arbitrary Theorem 2.7(a) implies that C has constant diameter r .

If $x \in C$, let $r_0 = d_+(C, x)$ so that $r \geq r_0 > 0$. Let $x_1 \in C$ with $|x - x_1| = r_0$. Thus, C is contained in the ball $V_{r_0}(x)$ and x_1 is on its boundary sphere. Let $u = (x_1 - x)/r_0$. ω_u achieves its maximum on $V_{r_0}(x)$ at x_1 and so achieves its maximum on C at x_1 . That is,

$G_C(u) = x_1$. By Theorem 3.12 (b) again $x_2 = x_1 - ru$ is an antipodal point for x_1 . Since $r_0 \leq r$, $x \in [x_1, x_2]$.

□

4. PARAMETRIZATIONS

Now let E be the Euclidean space \mathbb{R}^n with $n \geq 2$ and equipped with the usual metric.

First assume that C is a r -maximal subset of E , or, equivalently, a closed, convex set with constant diameter r .

In the previous section we defined the inverse Gauss map $G_C : S \rightarrow \text{Bdry}(C)$ which associates to each $u \in S$ the unique point of C at which ω_u achieves its maximum. By Proposition 3.10 this is a continuous, surjective function because an r -maximal set is strictly convex. By Theorem 3.12(b), if $x_1 = G_C(u)$ then $x_2 = x_1 - ru = G_C(-u)$ is the point of C antipodal to x_1 with $[x_1, x_2]$ parallel to u . Thus we have

$$(4.1) \quad u = (G_C(u) - G_C(-u))/r.$$

By Theorem 3.12 (a) as u varies over S , the pairs $G_C(u), G_C(-u)$ vary over all antipodal pairs. from Corollary 3.13 it follows that

$$(4.2) \quad C = \{tG_C(u) + (1-t)G_C(-u) : (u, t) \in S \times [0, 1]\}.$$

Now define the continuous function $H_C : S \rightarrow E$ by

$$(4.3) \quad H_C(u) \stackrel{\text{def}}{=} (G_C(u) + G_C(-u))/2 = G_C(u) - (r/2)u.$$

Hence,

$$(4.4) \quad H_C(-u) = H_C(u).$$

That is, H_C is an *even function*. Furthermore,

$$(4.5) \quad G_C(\pm u) = H_C(u) \pm (r/2)u.$$

Bayen, Lachand-Robert and Oudet call function H_C the *median surface function*. Following them, it will be the focus of our parametrization efforts. Imagine a stick of length r whose endpoints are the antipodal points. As the stick changes its position in C to point in direction u the midpoint is at $H_C(u)$.

Theorem 4.1. *If C is a r -maximal subset of E then for all $u, v \in S$*

$$(4.6) \quad (r/2)u \cdot u + u \cdot H_C(u) \geq (r/2)u \cdot v + u \cdot H_C(v).$$

or, equivalently,

$$(4.7) \quad u \cdot [H_C(v) - H_C(u)] \leq (r/4)|u - v|^2.$$

Proof: This equation is equivalent to

$$(4.8) \quad u \cdot G_C(u) \geq u \cdot G_C(v).$$

Because G_C is a surjective function this just says that ω_u achieves its maximum on $Bdry(C)$ at $x_1 = G_C(u)$. This follows from the definition (3.3) of G_C .

For the second equation we observe that $u \cdot u - u \cdot v = (1/2)|u - v|^2$ since $u, v \in S$.

□

Definition 4.2. Let $H : S \rightarrow E$ be a continuous, even function. We say that H satisfies the r -Median Inequality when for all $u, v \in S$

$$(4.9) \quad (r/2)u \cdot u + u \cdot H(u) \geq (r/2)u \cdot v + u \cdot H(v).$$

We say that H satisfies the Strict r -Median Inequality when the inequality is strict whenever $u \neq v$ in S .

Proposition 4.3. (a) The constant function $H = 0$ satisfies the Strict r -Median Inequality.

(b) Let $H_1, H_2 : S \rightarrow E$ be continuous, even functions with H_1 satisfying the r -Median Inequality. If H_2 satisfies the r -Median Inequality (or the Strict r -Median Inequality) then for $0 \leq \lambda < 1$ $\lambda H_1 + (1 - \lambda)H_2$ is a continuous, even function satisfying the r -Median Inequality (resp. the Strict r -Median Inequality).

Proof: If $u \neq v$ in S then $|u - v|^2 > 0$. Hence, the zero function satisfies the Strict r -Median Inequality. The second part is obvious.

□

Theorem 4.4. Assume that $H : S \rightarrow E$ is a continuous, even function. Define $C \subset E$ by

$$(4.10) \quad \begin{aligned} C &=_{def} \{ H(u) + t(r/2)u : (u, t) \in S \times [-1, 1] \} \\ &= \{ H(u) + t(r/2)u : (u, t) \in S \times [0, 1] \}. \end{aligned}$$

If H satisfies the r -Median Inequality then C is a r -maximal subset of E with $H_C = H$.

If H satisfies the Strict r -Median Inequality then, in addition, $G_C : S \rightarrow Bdry(C)$ is a homeomorphism and each point $x \in Bdry(C)$ has a unique antipodal point in $Bdry(C)$. In fact, $A_C(x) = G_C(-(G_C)^{-1}(x))$. Furthermore, at $x = G_C(u)$ there is a unique hyperplane of support for C which contains x and this hyperplane is perpendicular to u .

Proof: The two descriptions of C agree because H is even.

Define $G(u) = H(u) + (r/2)u$. Thus, the r -Median Inequality says $u \cdot G(u) \geq u \cdot G(v)$.

We begin by assuming that H satisfies the Strict r -Median Inequality.

If $v \neq u$ then $u \cdot G(u) > u \cdot G(v)$. Hence, $G(u) \neq G(v)$. That is, G is a homeomorphism of the sphere S onto its image $G(S) \subset E$. By the $n-1$ dimensional Jordan Curve Theorem, $G(S)$ is the boundary of the unique bounded component U of the open set $E \setminus G(S)$. Let $D = \overline{U} = U \cup G(S)$. On $G(S)$ the functional ω_u achieves its maximum at $G(u)$ by the r -Median Inequality. The open half-space $\{y \in E : u \cdot y > u \cdot J(u)\}$ is connected and disjoint from $G(S)$. Hence, it is disjoint from D . That is, $u \cdot G(u)$ is the maximum value of ω_u on D . It now follows from Proposition 2.12 that D is convex. Furthermore, $-u \cdot G(-u)$ is the maximum value of $\omega_{-u} = -\omega_u$ on D . Since, $G(u) - G(-u) = ru$ it follows that $\text{diam}_{\omega_u}(D) = r$. As $u \in S$ was arbitrary it follows that D is a closed, convex set of constant diameter r and so D is r -maximal. By definition of G_D we have $G_D(u) = G(u)$. Hence, $H_D(u) = (G_D(u) + G_D(-u))/2 = (G(u) + G(-u))/2 = H(u)$. In addition, $C = D$ by equation (4.2) because D is r -maximal with boundary $G(S)$. For each x in the boundary there is a unique $u \in S$ such that $G_C(u) = x$ and so its unique antipode is $G_C(-u)$. Furthermore, $u = v$ is the unique point $v \in S$ such that ω_v takes its maximum on C at x . Hence, there is a unique hyperplane of support at x and it is perpendicular to u . This completes the proof in the strict case.

Now assume that H satisfies r -Median Inequality and that $0 \leq \lambda < 1$.

By Proposition 4.3 $\lambda H = \lambda H + (1-\lambda)0$ satisfies the Strict r -Median Inequality. Let C_λ denote the 2-round subset defined by (4.10) with H replaced by λH .

Now we use the Hausdorff metric on compact subsets of E induced by the metric on E . As λ approaches 1 the compact sets C_λ approach C . It follows from Theorem 2.13 that C has constant diameter r . The proof that $G = G_C$ follows by letting λ approach 1 and then the proof that $H_C = H$ follows as before.

□

In the process we have proved the following.

Theorem 4.5. *Assume that $H : S \rightarrow E$ is a continuous, even function which satisfies the r -Median Inequality. For $\lambda \in [0, 1]$ define $C_\lambda \subset E$ by*

$$(4.11) \quad C_\lambda =_{\text{def}} \{ \lambda H(u) + t(r/2)u : (u, t) \in S \times [-1, 1] \}.$$

For $\lambda \in [0, 1)$ the function λH satisfies the Strict r -Median Inequality. As λ varies the r -maximal sets C_λ vary continuously with respect to the Hausdorff metric on the space of compacta in E . With $\lambda = 0$, C_λ is the $r/2$ ball centered at the origin in E .

Remark: Notice that the set of functions $H : S \rightarrow E$ which are even and satisfy the r -Median Inequality is closed under pointwise limits.

In order to obtain explicit examples where the r -Median Inequality holds we introduce smoothness conditions on H . With $G(u) = (r/2)u + H(u)$ defined on S the r -Median Inequality says that the function $u \cdot G(v)$ has a local maximum at $v = u$. Now assume that $H = (h_1, \dots, h_n)$ is at least C^1 . Precomposing H and G by the retraction $r : \mathbb{R}^n \setminus 0 \rightarrow S$ given by $r(x) = x/|x|$ we extend H and G and each of their components to become C^1 homogeneous functions of degree zero defined on $\mathbb{R}^n \setminus 0$. We will denote $\omega_u \circ G$ by N_u so that

$$(4.12) \quad N_u(x) = u \cdot [(r/2)x/|x| + H(x)] = (r/2)|x|^{-1}(u \cdot x) + \sum_i u_i h_i(x).$$

$$(4.13) \quad \frac{\partial N_u}{\partial x_j} = (r/2)[-|x|^{-3}(u \cdot x)x_j + |x|^{-1}u_j] + \sum_i u_i \frac{\partial h_i}{\partial x_j}.$$

The r -Median Inequality implies that $N_u = \omega_u \circ G$ has a local maximum at $x = u$. Because $x = u$ is a critical point these partials must vanish. This says that the image of the tangent map of H at u is contained in the subspace of vectors perpendicular to u , which we will denote u^\perp . That is, the image lies in the tangent hyperplane of the sphere S at u . Since this is true for every $u \in S$ and H is homogeneous of degree zero we obtain on $\mathbb{R}^n \setminus 0$

$$(4.14) \quad \sum_{i=1}^n x_i \cdot \frac{\partial h_i}{\partial x_j} = 0 \quad \text{for } j = 1, \dots, n.$$

Now define the real-valued function which is C^1 and homogeneous of degree 1:

$$(4.15) \quad g(x) =_{def} \sum_{i=1}^n x_i h_i(x).$$

The above equation is equivalent to:

$$(4.16) \quad \frac{\partial g}{\partial x_j} = h_j \quad \text{for } j = 1, \dots, n.$$

That is, H is the gradient of g . Since the gradient of g is C^1 it follows that g is C^2 .

From (4.16) we see that

$$(4.17) \quad g(x) =_{def} \sum_{i=1}^n x_i \frac{\partial g}{\partial x_i}.$$

We recall some well-known homogeneity results.

Lemma 4.6. *Let $g : \mathbb{R}^n \setminus 0 \rightarrow \mathbb{R}$ be differentiable. g is homogeneous of degree k iff*

$$(4.18) \quad kg(x) =_{def} \sum_{i=1}^n x_i \frac{\partial g}{\partial x_i}.$$

In that case, if g is odd (i.e. $g(-x) = -g(x)$) then the gradient is even. Conversely, if the gradient is even and $k \neq 0$ then g is odd.

Proof: Let $b(t) =_{def} g(tx)$ for $t > 0$. Then $b'(t) = \sum_{i=1}^n x_i \frac{\partial g}{\partial x_i}(tx)$.

If g is homogeneous of degree k then $b(t) = t^k b(x)$ and so $b'(t) = kt^{k-1}g(x)$. Set $t = 1$.

Conversely, if the equation holds then $tb'(t) = kb(t)$ and so $b(t) = t^k \cdot C$ for some constant C and setting $t = 1$ we see that the constant $C = g(x)$.

If $g(-x) = -g(x)$ then taking partials we get that each $\frac{\partial g}{\partial x_i}$ is even. The converse follows from the homogeneity equation (4.18) when $k \neq 0$. \square

Thus, a C^1 function H which is homogeneous of degree zero and whose tangent map at x has image in x^\perp for every x is exactly the gradient of a C^2 real-valued function g which is odd and homogeneous of degree one, i.e. which satisfies $g(tx) = tg(x)$ for all $t \neq 0$. Extending continuously, but not smoothly, to the origin we have $g(tx) = tg(x)$ for all real t .

Thus, equation (4.13) becomes

$$(4.19) \quad \frac{\partial N_u}{\partial x_j} = (r/2)[-|x|^{-3}(u \cdot x)x_j + |x|^{-1}u_j] + \sum_i u_i \frac{\partial^2 g}{\partial x_i \partial x_j}.$$

At $x = u$ the equation $\sum_i u_i \frac{\partial^2 g}{\partial x_i \partial x_j} = 0$ holds because $\frac{\partial g}{\partial x_j}$ is homogeneous of degree zero.

Definition 4.7. *Let $g : \mathbb{R}^n \setminus 0 \rightarrow \mathbb{R}$ be an odd, C^2 function which is homogeneous of degree $k = 1$. We say that g satisfies the Linear r -Median Condition if at every point $u \in S$ the eigenvalues of the*

Hessian matrix $(\frac{\partial^2 g}{\partial x_k \partial x_j})$ (hereafter denoted $\partial^2 g$) are contained in the closed interval $[-(r/2), (r/2)]$. We say that g satisfies the Strict r -Median Condition if at every point $u \in S$ the eigenvalues of $\partial^2 g$ are contained in the open interval $(-(r/2), (r/2))$.

Remark:(a) The Hessian matrix function $\partial^2 g$ on $\mathbb{R}^n \setminus 0$ is homogeneous of degree -1 . This is why the eigenvalue conditions are restricted to points u on the sphere. The eigenvalues will blow up as x approaches the origin.

(b) The function $\partial^2 g$ is odd and so if μ is an eigenvalue at u then $-\mu$ is an eigenvalue at $-u$.

Proposition 4.8. *Let $g : S \rightarrow \mathbb{R}$ be any odd, C^2 function on the unit sphere. Extend g to an odd, C^2 homogeneous function of degree 1 on $\mathbb{R}^n \setminus 0$ by $g(x) = |x|g(x/|x|)$. There is a maximum positive λ^* such that λ^*g satisfies the Linear r -Median Condition and then λg satisfies the Strict Linear r -Median Condition for all $\lambda \in [0, \lambda^*)$.*

Proof: Multiplying g by λ multiplies the eigenvalues of the Hessian at u by λ for every $u \in S$. As the eigenvalues vary continuously the result is obvious from compactness of S .

□

Lemma 4.9. *For $g : \mathbb{R}^n \setminus 0 \rightarrow \mathbb{R}$ an odd, C^2 function which is homogeneous of degree 1 let $H : \mathbb{R}^n \setminus 0 \rightarrow \mathbb{R}^n$ be the gradient of g . If H satisfies the r -Median Inequality, then g satisfies the Linear r -Median Condition.*

Proof: For any point $u \in S$ the vector u itself is an eigenvector of $\partial^2 g$ at u with eigenvalue zero. The eigenvectors with nonzero eigenvalues lie in u^\perp . Let $v \in u^\perp$ and define the path $p : [0, \pi] \rightarrow S$ by

$$(4.20) \quad \begin{aligned} p(t) &= \cos(t)u + \sin(t)v, \quad \text{so that} \\ p'(t) &= -\sin(t)u + \cos(t)v \end{aligned}$$

At $p(t)$, the vector $p(t)$ is an eigenvector of $\partial^2 g$ with eigenvalue zero and so at $p(t)$

$$(4.21) \quad \sum_i u_i \frac{\partial^2 g}{\partial x_i \partial x_j} = -\tan(t) \sum_i v_i \frac{\partial^2 g}{\partial x_i \partial x_j}.$$

Differentiating $N_u(p(t))$ we get, after applying (4.21) on each side of $\partial^2 g$

(4.22)

$$\begin{aligned} \frac{dN_u \circ p}{dt} &= (r/2)u \cdot p'(t) + \sum_{i,j} u_i \frac{\partial^2 g}{\partial x_i \partial x_j} p'(t)_j = \\ &= -(r/2) \sin(t) - \tan(t) \sum_{i,j} v_i \frac{\partial^2 g}{\partial x_i \partial x_j} [\sin(t) \tan(t) + \cos(t)] v_j \\ &= -\sin(t) [(r/2) + \sec(t)(\sin(t) \tan(t) + \cos(t)) \sum_{i,j} v_i \frac{\partial^2 g}{\partial x_i \partial x_j} v_j]. \end{aligned}$$

By the r -Median Inequality N_u has a local maximum at u . So by the Mean Value Theorem there is a positive sequence $\{t_k\}$ tending to zero such that $d(N_u \circ p)/dt \leq 0$ at each point $p(t_k)$. Applying (4.22), dividing by the positive number $\sin(t_k)$ and then letting k tend to infinity we obtain at $u = p(0)$:

$$(4.23) \quad (r/2) + \sum_{i,j} v_i \frac{\partial^2 g}{\partial x_i \partial x_j} v_j \geq 0$$

for all $v \in u^\perp$. In particular, every nonzero eigenvalue μ of $\partial^2 g$ at u satisfies $\mu \geq -(r/2)$. Since $-\mu$ is an eigenvalue at $-u$ we have $(r/2) \geq \mu \geq -(r/2)$ as required. \square

Theorem 4.10. *For $g : \mathbb{R}^n \setminus 0 \rightarrow \mathbb{R}$ an odd, C^2 function which is homogeneous of degree 1 let $H : \mathbb{R}^n \setminus 0 \rightarrow \mathbb{R}^n$ be the even, C^1 function which is the gradient of g so that H is homogeneous of degree 0.*

The function g satisfies the Strict Linear r -Median Condition iff $-r/2$ is not an eigenvalue of the Hessian matrix $(\frac{\partial^2 g}{\partial x_k \partial x_j})$ at any point u of S .

If g satisfies the Strict Linear r -Median Condition then H satisfies the Strict r -Median Inequality.

Furthermore, g satisfies the Linear r -Median Condition iff H satisfies the r -Median Inequality.

Proof: It is obvious that if g satisfies the Strict Linear r -Median Condition then $-r/2$ is not an eigenvalue of $\partial^2 g$ at any point of S .

Now assume that $-r/2$ is not an eigenvalue of $\partial^2 g$ at any point of S .

Let $\bar{G} = (r/2)I + H$ which agrees with G on S . For \bar{G} we are not projecting to the sphere for the $r/2$ times the identity term and so \bar{G} is not homogeneous. The assumption that $-r/2$ is never an eigenvalue

says exactly that the tangent map of \bar{G} is injective at each point v of the sphere. Meanwhile, on the sphere the tangent map of G at v maps v^\perp into itself. Since this is the restriction of the tangent map of \bar{G} as well, it follows that at each point $v \in S$ the tangent map of G is an isomorphism of v^\perp .

Now consider $N_u = \omega_u \circ G$ on S . At any point $v \neq \pm u$, ω_u maps v^\perp onto \mathbb{R} . Hence, $\pm u$ are the only critical points of N_u on S . On the other hand, the maximum and minimum values of N_u on S are certainly critical values. Since $\omega_u(G(u)) - \omega_u(G(-u)) = r > 0$ it follows that $N_u(u)$ and $N_u(-u)$ are the maximum and minimum values of N_u on S . Furthermore, no other points are critical and so N_u achieves its maximum only at u . This proves that H satisfies the Strict r -Median Inequality.

To complete the proof of the first part, we must show that g satisfies the Strict Linear r -Median Condition.

Let μ is an eigenvalue of $\partial^2 g$ at some point. In that case, Lemma 4.9 implies that μ lies in the interval $[-(r/2), (r/2)]$. By assumption μ is not $-(r/2)$. As $-\mu$ is an eigenvalue at the antipodal point, $-\mu$ is not $-(r/2)$ either. Hence, μ is in the open interval and the Strict Linear r -Median Condition holds.

Now if g satisfies the Linear r -Median Condition then for $\lambda < 1$, λg satisfies the Strict Linear r -Median Condition and so λH satisfies the Strict r -Median Inequality. Letting λ approach 1 we see that H satisfies the r -Median Inequality.

Conversely, if H satisfies the r -Median Inequality then g satisfies the Linear r -Median Condition by Lemma 4.9 again.

□

Theorem 4.11. *Let $g : \mathbb{R}^n \setminus 0 \rightarrow \mathbb{R}$ an odd, C^2 function which is homogeneous of degree 1. There exists a nonnegative real number r^* such that*

$$(4.24) \quad \left\{ \mu : \mu \text{ is an eigenvalue of the matrix } \partial^2 g \text{ at some } u \in S \right\} = [-(r^*/2), (r^*/2)].$$

Let H be the gradient of g and let r be a positive real number. If $r > r^$ then H satisfies the Strict r -Median Inequality. If $r = r^*$ then H satisfies the r -Median Inequality. If $r < r^*$ then H does not satisfy the r -Median Inequality.*

Proof: Since the eigenvalues of $\partial^2 g$ vary continuously on the compact set S it follows that the set of eigenvalues is compact as u varies over S . At every point $u \in S$, u is an eigenvector of $\partial^2 g$ with eigenvalue

0. Furthermore, if μ is an eigenvalue at u then $-\mu$ is an eigenvalue at $-u$. Hence, there is a nonnegative r^* such that the set of eigenvalues is contained in the closed interval $[-(r^*/2), (r^*/2)]$ and the endpoints, $\pm r^*/2$ and also 0 are eigenvalues. Now suppose $r > 0$ with $-r/2$ not an eigenvalue. Theorem 4.10 implies that the set of eigenvalues is contained in $(-(r/2), (r/2))$ and so $r > r^*$. Moreover, if $r/2$ is not an eigenvalue then by symmetry $-r/2$ is not an eigenvalue. Thus, the set of eigenvalues is the entire closed interval $[-(r^*/2), (r^*/2)]$.

Clearly, g satisfies the Linear r -Median Condition iff $r \geq r^*$ and the Strict Linear r -Median Condition iff $r > r^*$. Thus, the remaining results follow from Theorem 4.10.

□

Thus, any C^2 odd, homogeneous g satisfies the Linear r -Median Condition for r sufficiently large. We can use r as a parameter for the solid C described by (4.10), with H the gradient of g . At $r = 0$, C is just the image $H(S)$ and for small $r \geq 0$ the Linear r -Median Condition will not hold. The set C will not be convex. For $r \geq r^*$ sufficiently large, C becomes convex and the Linear r -Median Condition holds. So for all $r \geq r^*$ the set C is a solid of constant diameter r .

Notice that the trivial case of an odd function g of degree 1 is the linear function $a \cdot x$ for a constant vector a . Adding this linear function to g translates the gradient H by a and leaves the Hessian matrices unaffected. In particular the Linear r -Median Conditions are unaffected. The resulting r -maximal sets are translated in \mathbb{R}^n .

If O is an orthogonal $n \times n$ matrix, let $u \mapsto O(u)$ denote the associated linear map. If $g_O = g \circ O^{-1}$ then the gradient H_O satisfies $H_O(u) = O(H(O^{-1}(u)))$. Hence,

$$(4.25) \quad \begin{aligned} G_O(u) &= (r/2)u + H_O(u) = \\ &O((r/2)O^{-1}(u) + H(O^{-1}(u))) = O(G(O^{-1}(u))). \end{aligned}$$

Thus, the image of G_O is the image of G rotated by O .

In the case of the Euclidean Plane, i.e. $n = 2$, there is a somewhat different parametric approach.

We use polar coordinates ρ, θ for $\mathbb{R}^2 \setminus 0$. Thus, the angular coordinate θ parametrizes the unit circle S . A function $H = (h_1, h_2) : \mathbb{R}^2 \setminus 0 \rightarrow \mathbb{R}^2$ is homogeneous of degree 0 when it is a function of θ alone. A function $g : \mathbb{R}^2 \setminus 0 \rightarrow \mathbb{R}$ is homogeneous of degree 1 when there exists a function $a : \mathbb{R}^2 \setminus 0 \rightarrow \mathbb{R}$ homogeneous of degree 0 with $g(\rho, \theta) = \rho a(\theta)$. H is an

even function when $a(\theta + \pi) = a(\theta)$ for all θ , i.e. a has period π and H is an odd function when $a(\theta + \pi) = -a(\theta)$, i.e. a has *anti-period* equal to π . For example, $\cos(k\theta)$ and $\sin(k\theta)$ are both π periodic when k is even and π anti-periodic when k is odd.

With $x = \rho \cos(\theta)$, $y = \rho \sin(\theta)$ the coordinate change matrices are given by

$$(4.26) \quad \begin{pmatrix} \partial x / \partial \rho & \partial x / \partial \theta \\ \partial y / \partial \rho & \partial y / \partial \theta \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\rho \sin(\theta) \\ \sin(\theta) & \rho \cos(\theta) \end{pmatrix},$$

and its inverse

$$(4.27) \quad \begin{pmatrix} \partial \rho / \partial x & \partial \rho / \partial y \\ \partial \theta / \partial x & \partial \theta / \partial y \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\rho^{-1} \sin(\theta) & \rho^{-1} \cos(\theta) \end{pmatrix},$$

Now consider $g(\rho, \theta) = \rho a(\theta)$ with a a C^2 function of the angular coordinate. The gradient $H = (h_1, h_2)$ is given by:

$$(4.28) \quad h_1(\theta) = \frac{\partial g}{\partial \theta} \frac{\partial \theta}{\partial x} + \frac{\partial g}{\partial \rho} \frac{\partial \rho}{\partial x} = -a'(\theta) \sin(\theta) + a(\theta) \cos(\theta),$$

$$h_2(\theta) = \frac{\partial g}{\partial \theta} \frac{\partial \theta}{\partial y} + \frac{\partial g}{\partial \rho} \frac{\partial \rho}{\partial y} = a'(\theta) \cos(\theta) + a(\theta) \sin(\theta).$$

Now define β so that

$$(4.29) \quad (r/2)\beta(\theta) \stackrel{\text{def}}{=} a(\theta) + a''(\theta).$$

Regarding the restriction of H to S as a path in \mathbb{R}^2 we see that its velocity vector is given by

$$(4.30) \quad H'(\theta) = (h_1'(\theta), h_2'(\theta)) = (r/2)\beta(\theta)(-\sin(\theta), \cos(\theta)).$$

Finally, the Hessian matrix for g is given by

$$(4.31) \quad \begin{pmatrix} \partial^2 g / \partial x^2 & \partial^2 g / \partial x \partial y \\ \partial^2 g / \partial y \partial x & \partial^2 g / \partial y^2 \end{pmatrix} = \begin{pmatrix} \partial h_1 / \partial x & \partial h_1 / \partial y \\ \partial h_2 / \partial x & \partial h_2 / \partial y \end{pmatrix}$$

$$= \rho^{-1}(r/2)\beta(\theta) \begin{pmatrix} \sin^2(\theta) & -\sin(\theta) \cos(\theta) \\ -\sin(\theta) \cos(\theta) & \cos^2(\theta) \end{pmatrix}$$

The two eigenvalues of the latter matrix are 0 and $\rho^{-1}(r/2)\beta(\theta)$. From all this we obtain

Proposition 4.12. *Assume that $a : S \rightarrow \mathbb{R}$ is a π anti-periodic, C^{k+2} function. The function $\beta : S \rightarrow \mathbb{R}$ defined by $\beta(\theta) = (2/r)[a(\theta) + a''(\theta)]$*

is a π anti-periodic, C^k function such that

$$(4.32) \quad \int_0^\pi \beta(\theta)(-\sin(\theta), \cos(\theta))d\theta = (0, 0).$$

Furthermore, the odd function, homogeneous of degree 1, defined by $g(\rho, \theta) = \rho a(\theta)$ satisfies the Linear r -Median Condition (or the Strict Linear r -Median Condition) iff $|\beta(\theta)| \leq 1$ for all $\theta \in \mathbb{R}$ (resp. $|\beta(\theta)| < 1$ for all $\theta \in \mathbb{R}$).

Conversely, given a π anti-periodic C^k function $\beta(\theta)$ which satisfies (4.32) there exists a π anti-periodic, odd C^{k+1} function $a(\theta)$ such that $(r/2)\beta = a + a''$.

Proof: Equation (4.32) follows from (4.30) because integrating H' from 0 to π yields $H(\pi) - H(0)$ which equals 0 because H is even. It is clear that if a is π anti-periodic then $\beta = (2/r)[a + a'']$ is π anti-periodic. The Linear r -Median Conditions follow from (4.31) because with $r = 1$ the only nonzero eigenvalue at θ is $(r/2)\beta(\theta)$.

Conversely, given β we can use variation of parameters to solve the differential equation:

$$(4.33) \quad a(\theta) =_{def} \int_0^\theta (r/2)\beta(s)[\cos(s)\sin(\theta) - \sin(s)\cos(\theta)] ds.$$

By (4.32) $a(0) = 0 = a(\pi)$. Because $\sin(\theta)$ and $\cos(\theta)$ are π anti-periodic it follows that

$$(4.34) \quad a(\theta + \pi) = - \int_0^{\theta+\pi} (r/2)\beta(s)[\cos(s)\sin(\theta) - \sin(s)\cos(\theta)] ds.$$

On the other hand, $\beta(s)(-\sin(s), \cos(s))$ is π periodic. Hence, in (4.32) the integral from θ to $\theta + \pi$ is zero for any θ . It follows that $a(\theta + \pi) = -a(\theta)$. Thus, a is π anti-periodic.

The general solution of the differential equation is obtained by adding $C_1 \cos(\theta) + C_2 \sin(\theta)$ for arbitrary constants C_1, C_2 . Adding such a term corresponds to adding a linear function to $g(r, \theta)$ and has the effect of translating the gradient H by a constant.

□

Above we used g or, equivalently, $a(\theta)$ to construct our parametrization. Now we will use $\beta(\theta)$ instead. As we are dealing with curves in the plane we will switch to the usual parametric representation for such curves. For example, $t \mapsto \mathbf{U}(t) =_{def} (\cos(t), \sin(t))$ parametrizes the unit circle with unit tangent vector $\mathbf{T}(t) =_{def} (-\sin(t), \cos(t))$. We also return now to the general case of r -rotundity.

We begin with a C^2 function $\beta : \mathbb{R} \rightarrow \mathbb{R}$ such that

- (i) $|\beta(t)| \leq 1$.
- (ii) $\beta(t + \pi) = -\beta(t)$.
- (iii) $\int_0^\pi \beta(u)(-\sin(u), \cos(u)) du = (0, 0)$.

The associated boundary curve (the one dimensional version of $G(u)$ above) has position and velocity vectors given by

$$(4.35) \quad \begin{aligned} \mathbf{R}(t) &=_{def} \int_0^t (r/2)(1 + \beta(u))(-\sin(u), \cos(u)) du \\ \mathbf{R}'(t) &= (r/2)(1 + \beta(t))(-\sin(t), \cos(t)). \end{aligned}$$

Thus, the unit tangent vector is $\mathbf{T}(t)$ and the speed is given by

$$(4.36) \quad \frac{ds}{dt} = (r/2)(1 + \beta(t)).$$

Notice that $\mathbf{T}'(t) = -\mathbf{U}(t)$ and so $-\mathbf{U}(t)$ is the unit normal vector. Hence,

$$(4.37) \quad \frac{d\mathbf{T}}{ds} = -[(r/2)(1 + \beta(t))]^{-1}\mathbf{U}(t)$$

By Proposition 4.12, Theorem 4.10, and Theorem 4.5 the closed curve bounds the r -maximal set in the plane:

$$(4.38) \quad C_\beta =_{def} \{u\mathbf{R}(t) + (1-u)\mathbf{R}(t+\pi) : (t, u) \in [0, 2\pi] \times [0, 1]\}$$

Notice that for the arclength of the closed curve we get

$$(4.39) \quad \int_0^{2\pi} (r/2)(1 + \beta(t)) dt = \pi r.$$

because the function β is π anti-periodic. This is the smooth special case of a general theorem due to Barbier, for its proof, see Lyusternik (1966) Section 12.

Theorem 4.13. *Let $\beta : [0, \pi) \rightarrow [-1, 1]$ be a measurable function such that $\int_0^\pi \beta(u)(-\sin(u), \cos(u)) du = (0, 0)$. Extend β to $[0, 2\pi)$ by $\beta(t + \pi) = -\beta(t)$ for $t \in [0, \pi)$. Then extend to \mathbb{R} to make the function 2π periodic. The Fourier series has nonzero coefficients only for $\sin(kt)$ and $\cos(kt)$ with k odd and greater than 1.*

The curve $\mathbf{R}(t) = \int_0^t (r/2)(1 + \beta(u))(-\sin(u), \cos(u)) du$ is continuous and bounds an r -maximal subset C_β .

Proof: One can construct a sequence of smooth functions β_n which satisfy conditions (i), (ii) and (iii) and which converge to β in $L^1([0, 2\pi])$. The characteristic functions of $[0, t]$, $\sin(t)$ and $\cos(t)$ all have norm at most 1 in L^∞ . Hence, the curves for β_n converge uniformly to the curve for β . The latter is continuous but need not be differentiable. Each

C_{β_n} is an r -maximal set and as $n \rightarrow \infty$ these sets converge to C_β with respect to the Hausdorff metric. By Theorem 2.13 C_β is r -maximal.

□

For the regular polygon with $2k+1$ vertices we use the function $\beta(t)$

$$(4.40) \quad \beta(t) = (-1)^i \quad \text{for } t \in \left[\frac{i\pi}{2k+1}, \frac{(i+1)\pi}{2k+1}\right) \quad \text{with } i = 0, \dots, 2k,$$

and extend over $(\pi, 2\pi)$ so that condition (ii) holds. Notice that for $t \in [0, \pi/2]$ we have

$$(4.41) \quad \beta(\pi - t) = \beta(t).$$

So it is clear that $\int_0^\pi \beta(t) \cos(t) dt = 0$. The sine condition is not obvious. It is true since the example works. However, there is a geometric argument.

$$(4.42) \quad \int_0^\pi \beta(t) \mathbf{T}(t) dt = \sum_{i=0}^{2k} (-1)^i \left[\mathbf{U}\left(\frac{(i+1)\pi}{2k+1}\right) - \mathbf{U}\left(\frac{i\pi}{2k+1}\right) \right].$$

For i odd we replace $-\left[\mathbf{U}\left(\frac{(i+1)\pi}{2k+1}\right) - \mathbf{U}\left(\frac{i\pi}{2k+1}\right)\right]$ by $\left[\mathbf{U}\left(\frac{(i+1)\pi}{2k+1} + \pi\right) - \mathbf{U}\left(\frac{i\pi}{2k+1} + \pi\right)\right] = \left[\mathbf{U}\left(\frac{(2k+1+i+1)\pi}{2k+1}\right) - \mathbf{U}\left(\frac{(2k+1+i)\pi}{2k+1}\right)\right]$ and so we have

$$(4.43) \quad \int_0^\pi \beta(t) \mathbf{T}(t) dt = \sum_{i=0}^{2k} \left[\mathbf{U}\left(\frac{(2i+1)\pi}{2k+1}\right) - \mathbf{U}\left(\frac{2i\pi}{2k+1}\right) \right].$$

This equals zero because the sum is invariant under rotation by $\frac{2\pi}{2k+1}$.

With $k = 1$ we have the triangle case.

In general, let μ be the measure on $[0, \pi/2]$ with density $\sin(t)$ so that $\mu([0, \pi/2]) = 1$. For $\beta(t)$ defined on $[0, \pi/2]$ with $|\beta(t)| \leq 1$ and such that $\int_0^{\pi/2} \beta(t) \mu(dt) = 0$ we can extend to $[\pi/2, \pi)$ so that (4.41) holds. Then extend to $(0, 2\pi)$ to get condition (ii). For example, if A is any measurable subset of $[0, \pi/2]$ with $\mu(A) = \frac{1}{2}$ then we can use $\beta(t) = \chi_A(t) - \chi_{A'}(t)$ with $A' = [0, \pi/2] \setminus A$. The interesting case here is with A a Cantor set.

Any curve with nonvanishing curvature can be parametrized by using the unit normal. This is the one dimensional version of the classical Gauss map.

Theorem 4.14. *Given a plane curve with radius of curvature bounded by r at every point then any sufficiently short piece can be embedded in the boundary of some r -maximal subset of the plane. To be precise, if the curve is parametrized by $t = \text{the angle } \theta \text{ of the unit normal and}$*

the parameter moves through an interval of length at most $\pi/3$ then the curve can be embedded within the boundary of an r -maximal subset.

Proof: By rotating we can assume that the angle t moves from 0 to some $\theta^* \leq \pi/3$. We can reverse the construction to define the associated function β on $[0, \theta^*]$.

Since the curve is parametrized by the angle of the unit normal, the velocity vector is a multiple of $\mathbf{T}(t)$. We define $\beta(t)$ so that $(r/2)(1 + \beta(t))$ is the speed at t . As above, the radius of curvature is given by $(r/2)(1 + \beta(t))$ and since this is between 0 and r we have $|\beta(t)| \leq 1$. Since $\mu([0, \pi/3]) = \frac{1}{2}$, we can extend β to $(\theta^*, \pi/2)$ by a suitable constant in $[-1, 1]$ so that $\int_0^{\pi/2} \beta(t)\mu(dt) = 0$ and proceed as above.

□

Remark: The $\pi/3$ restriction is needed. Notice that you can't use an arc of a circle with radius r and angle greater than $\pi/3$ because then the chord is longer than the radius.

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